Supertwistors, Super Yang-Mills Theory and Integrability

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Contents

Motivation for Twistor String Theory

- Witten's original motivation for twistor string theory was to find some new kind of gauge/string duality, i.e., some sort of weak-weak duality, contrary to Maldacena's AdS/CFT correspondence, which is of weak-strong type.

  [E. Witten, hep-th/0312171]

- Remember, this correspondence states the equivalence of $\mathcal{N} = 4$ SYM theory on compactified Minkowski space and type IIB superstring theory on $AdS_5 \times S^5$.

  [J. Maldacena, hep-th/9711200]

Motivation for Twistor String Theory

- To describe weakly coupled gauge theory in this setup, one needs to consider the full string theory (and vice versa).
  Makes it difficult to test the correspondence!
- $AdS_5 \times S^5$ has a $PSU(2,2|4)$ symmetry:

$$PSU(2,2|4) \sim \begin{pmatrix} \text{Bose} & \text{Fermi} \\ \text{Fermi} & \text{Bose} \end{pmatrix}$$

- Witten suggested to take the supertwistor space $\mathbb{C}P^{3|4}$ as the target space for a yet to be determined string theory:

$$(Z^1, \ldots, Z^4 | \psi^1, \ldots, \psi^4) \sim (\lambda Z^1, \ldots, \lambda Z^4 | \lambda\psi^1, \ldots, \lambda\psi^4)$$

for $\lambda \in \mathbb{C}^*$, since ...

Motivation for Twistor String Theory

- Consider $\mathbb{C}^{4|4}$. Its full supergroup of linear transformations is called $GL(4|4)$. Thus, $\mathbb{C}P^{3|4}$ has the symmetry group

$$PGL(4|4) \;=\; GL(4|4)/\{\text{ center }\}.$$

- On $\mathbb{C}^{4|4}$, we may consider

$$\Omega_0 \;\equiv\; \mathrm{d}Z^1 \wedge \cdots \wedge \mathrm{d}Z^4 \mathrm{d}\psi^1 \cdots \mathrm{d}\psi^4.$$

  The subgroup of $PGL(4|4)$ that preserves Ω_0 is $PSL(4|4)$.
- Remember that the superconformal group in 4*D* is just a real form of $PSL(4|4)$.
- Ω_0 is a section of the Berezinian of $T\mathbb{C}^{4|4}$, i.e., an integral form rather than a differential form.

Motivation for Twistor String Theory

- Furthermore, Ω_0 is invariant under

$$Z \mapsto \lambda Z \qquad \text{and} \qquad \psi \mapsto \lambda\psi,$$
$$\mathrm{d}Z \mapsto \lambda \mathrm{d}Z \qquad \text{and} \qquad \mathrm{d}\psi \mapsto \lambda^{-1}\mathrm{d}\psi.$$

- Thus, Ω_0 descends to a holomorphic measure Ω on $\mathbb{C}P^{3|4}$.
  $\mathbb{C}P^{3|4}$ is a Calabi-Yau supermanifold.
- The CY condition enables us to define a topological B-model with target $\mathbb{C}P^{3|4}$.

Motivation for Twistor String Theory

- Recall that the open topological B-model describes holomorphic bundles and their moduli, while the closed string sector describes variations of the complex structure of the target.
- In the (open) B-model we start with a stack of N space-filling D-branes on $\mathbb{C}P^{3|4}$. The gauge group will be $GL(N, \mathbb{C})$ and the basic field is a gauge-field $\mathcal{A}$ on $\mathbb{C}P^{3|4}$, which is of type $(0, 1)$ together with the action

$$S = \int \Omega \wedge \mathrm{tr}(\mathcal{A} \wedge \bar{\partial}\mathcal{A} + \tfrac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}),$$

i.e., holomorphic Chern-Simons theory.

Motivation for Twistor String Theory

- Via the Penrose-Ward transform (cf. below), we get the spectrum and parts of the interactions of $\mathcal{N} = 4$ SYM theory.

[R. Penrose, Rept. Math. Phys. 12 (1977) 65]

[R. S. Ward, Phys. Lett. A 61 (1977) 81]

- Witten then showed that the interactions of the full gauge theory come from $D1$-instantons on which open strings can end, i.e., he demonstrated that perturbative gauge theory (at tree-level) can be described as a $D1$-instanton expansion of the B-model.

Flag Manifolds

- For the moment, let's forget about the word super and restrict ourselves to ordinary twistor geometry. To add fermions later on won't be a big deal.
- Let V be a complex vector space of dimension n and consider its flag manifold

$$F_{d_1\cdots d_m}(V) \equiv \{(S_1,\ldots,S_m)\,|\,S_i\subset V,\ \dim_{\mathbb{C}}S_i=d_i,\ S_1\subset S_2\subset\cdots\subset S_m\}.$$

  - Examples: $F_1=\mathbb{C}P^{n-1}$ and $F_k=G_{k,n}(\mathbb{C})$.

Double Fibration

- Let $\mathbb{T}$ be a fixed complex vector space of dimension 4 and call it twistor space.
- Then, we've the natural double fibration

$$\begin{array}{ccc} & F_{12}(\mathbb{T}) & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ F_1(\mathbb{T}) & & F_2(\mathbb{T}) \end{array}$$

with $\pi_1(S_1, S_2) = S_1$ and $\pi_2(S_1, S_2) = S_2$.

- Define:
  - $\mathbb{P} \equiv F_1(\mathbb{T}) = \mathbb{C}P^3$ – projective twistor space
  - $\mathbb{M} \equiv F_2(\mathbb{T}) = G_{2,4}(\mathbb{C})$ – compactified complexified 4-space
  - $\mathbb{F} \equiv F_{12}(\mathbb{T})$ – correspondence space

Coordinates

- Then, we've the following proposition (cf. below):

$$\begin{array}{ccc} \text{point in } \mathbb{P} & \longleftrightarrow & \mathbb{C}P^2 \subset \mathbb{M} \\ \mathbb{C}P^1 \subset \mathbb{P} & \longleftrightarrow & \text{point in } \mathbb{M} \end{array}$$

- Let

$$x = (x^{\alpha\dot{\alpha}}) \in \mathbb{C}^{2\times 2} \overset{\varphi}{\mapsto} \begin{bmatrix} x \\ \mathbb{1}_2 \end{bmatrix}$$

be a coordinate mapping for $\mathbb{M}$. Then we define the coordinate chart on $\mathbb{M}$ by

$$\mathcal{M} \equiv \varphi(\mathbb{C}^{2\times 2}) \cong \mathbb{C}^4.$$

We call $\mathcal{M}$ affine complexified 4-space, noting that it is simply one of six choices of standard coordinate charts.

Coordinates

- Define the affine parts of $\mathbb{P}$ and $\mathbb{F}$ according to:
  - $\mathcal{P} \equiv \pi_1 \circ \pi_2^{-1}(\mathcal{M})$
  - $\mathcal{F} \equiv \pi_2^{-1}(\mathcal{M})$
- Then, we've $\mathcal{F} \cong \mathcal{M} \times \mathbb{C}P^1$.

Proof: Let $\lambda = [\lambda_{\dot{1}}, \lambda_{\dot{2}}]$ be homogeneous coordinates of $\mathbb{C}P^1$ and x as above. Consider:

$$(x, [\lambda]) \mapsto \left(\begin{bmatrix} x \\ \mathbb{1}_2 \end{bmatrix} \lambda, \begin{bmatrix} x \\ \mathbb{1}_2 \end{bmatrix} \right) = \left(\begin{bmatrix} x\lambda \\ \lambda \end{bmatrix}, \begin{bmatrix} x \\ \mathbb{1}_2 \end{bmatrix} \right) = \left(S_1^{x,\lambda}, S_2^{x,\lambda} \right) \in \mathbb{F}.$$

This defines an diffeomorphism. ∎

Coordinates

- Thus, our double fibration in terms of these coordinates has the form

$$\begin{array}{ccc} & (x,[\lambda]) \in \mathbb{F} & \\ {\scriptstyle \pi_1} \swarrow & & \searrow {\scriptstyle \pi_2} \\ [x\lambda,\lambda] \in \mathbb{P} & & x \in \mathbb{M} \end{array}$$

- In particular, this shows that $\mathcal{P}$ is the total space of the rank 2 holomorphic vector bundle

$$\mathcal{O}(1) \oplus \mathcal{O}(1) \;\to\; \mathbb{C}P^1.$$

Coordinates

- In the following, we're only interested in the "affinization" of our initial double fibration and consider:

$$\begin{array}{ccc} & \mathcal{F}^5 = \mathcal{M}^4 \times \mathbb{C}P^1 & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathcal{P}^3 & & \mathcal{M}^4 \end{array}$$

  Here, we've indicated the respective dimensions and introduce the coordinates:

  - $\mathcal{M}^4$: $x^{\alpha\dot{\alpha}}$
  - $\mathcal{F}^5$: $x^{\alpha\dot{\alpha}}$ and $\lambda_\pm$
  - $\mathcal{P}^3$: $z_+^\alpha = x^{\alpha\dot{\alpha}}\lambda^+_{\dot{\alpha}} = x^{\alpha\dot{1}} + x^{\alpha\dot{2}}\lambda_+$,
    $z_+^\alpha = \lambda_+ z_-^\alpha$ and $z_\pm^3 = \lambda_\pm$.

The Word Super

- Next, let's add fermionic degrees of freedom, i.e., consider the extension:

$$\begin{array}{ccc} & \mathcal{F}^{5|2\mathcal{N}} = \mathcal{M}^{4|2\mathcal{N}} \times \mathbb{C}P^1 & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ \mathcal{P}^{3|\mathcal{N}} & & \mathcal{M}^{4|2\mathcal{N}} \end{array}$$

  The additional coordinates on $\mathcal{M}^{4|2\mathcal{N}}$ are $\eta_i^{\dot\alpha}$ and they correspond on $\mathcal{P}^{3|\mathcal{N}}$ to $\eta_i^{\pm} = \eta_i^{\dot\alpha}\lambda_{\dot\alpha}^{\pm}$.
- Thus, $\mathcal{P}^{3|\mathcal{N}}$ is nothing but

$$\mathcal{O}(1) \otimes \mathbb{C}^2 \oplus \Pi\mathcal{O}(1) \otimes \mathbb{C}^{\mathcal{N}} \ \to\ \mathbb{C}P^1.$$

- Note that the 1st Chern number of $\mathcal{P}^{3|\mathcal{N}}$ is $c_1 = 4 - \mathcal{N}$.

Reality

- Now one may define real structures to get the signatures $++++$ and $--++$, respectively.
- In these cases, our double fibration reduces to a single fibration, since we have the diffeomorphism:

$$\mathcal{P}^{3|\mathcal{N}} \;\cong\; \mathcal{F}^{6|2\mathcal{N}} \;\cong\; \mathbb{R}^{4|2\mathcal{N}} \times \mathbb{C}P^1$$

[A. D. Popov, C. Sämann, hep-th/0405123]

- In the following, we shall restrict ourselves to the Euclidean setting.

Self-Dual SYM Theory

- Let's now discuss supergauge theory in the supertwistor context.
- Consider a holomorphic vector bundle $\mathcal{E} \to \mathcal{P}^{3|\mathcal{N}}$ which is characterized by the transition function f_{+-}.
- Then, $D^{\pm}_{\alpha} = \lambda^{\dot\alpha}_{\pm}\partial_{\alpha\dot\alpha}$, $\partial_{\bar\lambda_{\pm}}$ and $D^i_{\pm} = \lambda^{\dot\alpha}_{\pm}\partial^i_{\dot\alpha}$ form a basis of $T^{0,1}\mathcal{P}^{3|\mathcal{N}}$ and annihilate f_{+-}.
- Assume further that $\mathcal{E}$ is holomorphically trivial on any $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}}$. Then, Birkhoff tells us that

$$f_{+-} = \psi_+^{-1}\psi_-, \qquad \partial_{\bar\lambda_{\pm}}\psi_{\pm} = 0.$$

Self-Dual SYM Theory

- Therefore, we learn that $\psi_+ D^+_\alpha \psi_+^{-1} = \psi_- D^+_\alpha \psi_-^{-1}$ and $\psi_+ D^i_+ \psi_+^{-1} = \psi_- D^i_+ \psi_-^{-1}$ must be at most linear in λ_+.
- Thus, we introduce a Lie algebra valued one-form $\mathcal{A}$ with components:

$$\begin{aligned} \mathcal{A}^+_\alpha &:= \lambda^{\dot\alpha}_+ \mathcal{A}_{\alpha\dot\alpha} = \psi_\pm D^+_\alpha \psi_\pm^{-1}, \\ \mathcal{A}_{\bar\lambda_\pm} &:= 0, \\ \mathcal{A}^i_+ &:= \lambda^{\dot\alpha}_+ \mathcal{A}^i_{\dot\alpha} = \psi_\pm D^i_+ \psi_\pm^{-1} \end{aligned}$$

- In summary, we find the linear system

$$(D^+_\alpha + \mathcal{A}^+_\alpha)\psi_\pm = 0, \quad \partial_{\bar\lambda_\pm}\psi_\pm = 0, \quad (D^i_+ + \mathcal{A}^i_+)\psi_\pm = 0.$$

Self-Dual SYM Theory

- Clearly, we have certain compatibility conditions which are:

$$
\begin{aligned}
[\nabla_{\alpha\dot\alpha},\nabla_{\beta\dot\beta}]+[\nabla_{\alpha\dot\beta},\nabla_{\beta\dot\alpha}] &= 0,\\
[\nabla^i_{\dot\alpha},\nabla_{\beta\dot\beta}]+[\nabla^i_{\dot\beta},\nabla_{\beta\dot\alpha}] &= 0,\\
\{\nabla^i_{\dot\alpha},\nabla^j_{\dot\beta}\}+\{\nabla^i_{\dot\beta},\nabla^j_{\dot\alpha}\} &= 0,
\end{aligned}
$$

where

$$
\nabla_{\alpha\dot\alpha} := \partial_{\alpha\dot\alpha}+\mathcal{A}_{\alpha\dot\alpha}, \qquad \nabla^i_{\dot\alpha} := \partial^i_{\dot\alpha}+\mathcal{A}^i_{\dot\alpha}.
$$

Self-Dual SYM Theory

- Recall that $\mathcal{N} = 4$ self-dual SYM theory has:
  - a self-dual gauge potential $A_{\alpha\dot{\alpha}}$,
  - 4 positive chirality spinors χ^i_α,
  - 6 scalars $W^{ij} = -W^{ji}$,
  - 4 negative chirality spinors $\chi_{i\dot{\alpha}}$,
  - an anti-self-dual two-form $G_{\dot{\alpha}\dot{\beta}}$.
- Imposing the transversal gauge $\eta^{\dot{\alpha}}_i \mathcal{A}^i_{\dot{\alpha}} = 0$, we find the superfield expansions

$$
\begin{aligned}
\mathcal{A}_{\alpha\dot{\alpha}} &= A_{\alpha\dot{\alpha}} + \epsilon_{\dot{\alpha}\dot{\beta}} \chi^i_\alpha \eta^{\dot{\beta}}_i + \cdots \\
\mathcal{A}^i_{\dot{\alpha}} &= \epsilon_{\dot{\alpha}\dot{\beta}} W^{ij} \eta^{\dot{\beta}}_j + \tfrac{4}{3} \epsilon^{ijkl} \epsilon_{\dot{\alpha}\dot{\beta}} \chi_{k\dot{\gamma}} \eta^{\dot{\gamma}}_l \eta^{\dot{\beta}}_j - \\
&\quad - \tfrac{5}{6} \epsilon^{ijkl} \epsilon_{\dot{\alpha}\dot{\beta}} (G_{\dot{\gamma}\dot{\delta}} \delta^m_l + \cdots) \eta^{\dot{\gamma}}_k \eta^{\dot{\delta}}_m \eta^{\dot{\beta}}_j + \cdots
\end{aligned}
$$

Self-Dual SYM Theory

- Altogether, we obtain the e.o.m. of $\mathcal{N}=4$ self-dual SYM theory on $\mathbb{R}^4$:

$$
\begin{aligned}
f_{\dot\alpha\dot\beta} &= 0,\\
\epsilon^{\dot\alpha\dot\beta}\nabla_{\alpha\dot\alpha}\chi^i_{\beta} &= 0,\\
\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}W^{ij}+\epsilon^{\alpha\beta}\{\chi^i_\alpha,\chi^j_\beta\} &= 0,\\
\epsilon^{\dot\alpha\dot\beta}\nabla_{\alpha\dot\alpha}\chi_{i\dot\beta}-\tfrac{1}{2}\epsilon_{ijkl}[W^{kl},\chi^j_\alpha] &= 0,\\
\epsilon^{\dot\alpha\dot\beta}\nabla_{\alpha\dot\alpha}G_{\dot\beta\dot\gamma}+\{\chi^i_\alpha,\chi_{i\dot\gamma}\}+\tfrac{1}{4}\epsilon_{ijkl}[\nabla_{\alpha\dot\gamma}W^{ij},W^{kl}] &= 0.
\end{aligned}
$$

PW Transform

- From $\mathcal{A}^+_\alpha = \psi_\pm D^+_\alpha \psi_\pm^{-1}$ and $\mathcal{A}^i_+ = \psi_\pm D^i_+ \psi_\pm^{-1}$ we deduce

$$\mathcal{A}_{\alpha\dot\alpha} = \oint_{S^1} \frac{\mathrm{d}\lambda_+}{2\pi\mathrm{i}\lambda_+} \frac{\mathcal{A}^+_\alpha}{\lambda^{\dot\alpha}_+}, \quad \mathcal{A}^i_{\dot\alpha} = \oint_{S^1} \frac{\mathrm{d}\lambda_+}{2\pi\mathrm{i}\lambda_+} \frac{\mathcal{A}^i_+}{\lambda^{\dot\alpha}_+}.$$

Penrose-Ward Transform

- Furthermore, when $\psi_\pm \mapsto \psi_\pm h_\pm$ (where $h_\pm$ holomorphic) then $(\mathcal{E}, f_{+-}) \mapsto (\mathcal{E}', h_+^{-1} f_{+-} h_-)$, i.e., $\mathcal{E} \sim \mathcal{E}'$. Under such transformations, the components of $\mathcal{A}$ do not change. On the other hand, gauge transformations of $\mathcal{A}$ are induced by $\psi_\pm \mapsto g^{-1}\psi_\pm$ which leaves f_{+-} invariant.

Supertwistor Correspondence I

- Thus, we've decribed a one-to-one correspondence between equivalence classes of holomorphic vector bundles over the supertwistor space $\mathcal{P}^{3|\mathcal{N}}$ which are holomorphically trivial along any $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}}$ and gauge equivalence classes of solutions to the $\mathcal{N}$-extended self-dual SYM equations on $\mathbb{R}^4$.

Another Trivialization

- Above, we've chosen a special trivialization of $\mathcal{E}$. However, one can chose more general trivializations, s.t.

$$f_{+-} \;=\; \psi_+^{-1}\psi_- \;=\; \hat{\psi}_+^{-1}\hat{\psi}_- .$$

- Let's choose the following one:

$$f_{+-} \;=\; \hat{\psi}_+^{-1}\hat{\psi}_-, \qquad D^i_\pm\hat{\psi}_\pm \;=\; 0$$

- Thus, $\varphi := \psi_+\hat{\psi}_+^{-1} = \psi_-\hat{\psi}_-^{-1}$ is globally well defined and induces a gauge transformation of $\mathcal{A}$ according to $\hat{\psi}_\pm \mapsto \varphi^{-1}\psi_\pm$, s.t.

$$(D^+_\alpha + \hat{\mathcal{A}}^+_\alpha)\hat{\psi}_\pm \;=\; 0, \quad (\partial_{\bar{\lambda}_\pm} + \hat{\mathcal{A}}_{\bar{\lambda}_\pm})\hat{\psi}_\pm \;=\; 0, \quad D^i_\pm\hat{\psi}_\pm \;=\; 0.$$

HCS Theory

- The compatibility conditions of the previous system are given by (suppressing the "$+$"):

$$\begin{aligned} D_\alpha \hat{\mathcal{A}}_\beta - D_\beta \hat{\mathcal{A}}_\alpha + [\hat{\mathcal{A}}_\alpha, \hat{\mathcal{A}}_\beta] &= 0, \\ \partial_{\bar\lambda} \hat{\mathcal{A}}_\alpha - D_\alpha \hat{\mathcal{A}}_{\bar\lambda} + [\hat{\mathcal{A}}_{\bar\lambda}, \hat{\mathcal{A}}_\alpha] &= 0, \end{aligned}$$

  i.e., hCS theory on $\mathcal{P}^{3|\mathcal{N}}$.
- What's the explicit form of $\hat{\mathcal{A}}_\alpha$ and $\hat{\mathcal{A}}_{\bar\lambda}$?
  Recall that $\hat{\mathcal{A}}_\alpha$ and $\hat{\mathcal{A}}_{\bar\lambda}$ are sections of $\mathcal{O}(1)$ and $\overline{\mathcal{O}}(-2)$.
  This, together with the fact that the η_i are $\Pi\mathcal{O}(1)$ valued fixes the λ-dependence of the components of $\hat{\mathcal{A}}$ (up to gauge transformations).

HCS Theory

- We find:

$$\begin{aligned}\hat{\mathcal{A}}_\alpha &= \lambda^{\dot\alpha} A_{\alpha\dot\alpha} + \eta_i \chi^i_\alpha + \tfrac{1}{2!}\gamma\,\eta_i\eta_j\hat\lambda^{\dot\alpha} W^{ij}_{\alpha\dot\alpha} + \\ &\quad + \tfrac{1}{3!}\gamma^2\eta_i\eta_j\eta_k\hat\lambda^{\dot\alpha}\hat\lambda^{\dot\beta}\chi^{ijk}_{\alpha\dot\alpha\dot\beta} + \tfrac{1}{4!}\gamma^3\eta_i\eta_j\eta_k\eta_l\hat\lambda^{\dot\alpha}\hat\lambda^{\dot\beta}\hat\lambda^{\dot\gamma} G^{ijkl}_{\alpha\dot\alpha\dot\beta\dot\gamma},\\ \hat{\mathcal{A}}_{\bar\lambda} &= \tfrac{1}{2!}\gamma^2\eta_i\eta_j W^{ij} + \tfrac{1}{3!}\gamma^3\eta_i\eta_j\eta_k\hat\lambda^{\dot\alpha}\chi^{ijk}_{\dot\alpha} + \\ &\quad + \tfrac{1}{4!}\gamma^4\eta_i\eta_j\eta_k\eta_l\hat\lambda^{\dot\alpha}\hat\lambda^{\dot\beta} G^{ijkl}_{\dot\alpha\dot\beta},\end{aligned}$$

where the red colored fields represent the field content of $\mathcal{N}=4$ self-dual SYM theory.

HCS Theory

- Substituting these expansions into the field equations, we get the e.o.m. of $\mathcal{N} = 4$ self-dual SYM theory on $\mathbb{R}^4$.
- What about the action?
  Recall that $\mathcal{P}^{3|4}$ is a CY, i.e., we can write down an action functional for hCS theory,

$$S = \int_{\mathcal{Y}} \Omega \wedge \mathrm{tr}(\mathcal{A} \wedge \bar{\partial}\mathcal{A} + \tfrac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}),$$

  where $\mathcal{Y} \subset \mathcal{P}^{3|4}$ is given by $\bar{\eta}_i = 0$.
- Our field expansions plus integration reproduce the Lorentz invariant Siegel action for $\mathcal{N} = 4$ self-dual SYM theory.
  [W. Siegel, hep-th/9205075]

Supertwistor Correspondence II

- Thus, once again we've decribed a one-to-one correspondence between equivalence classes of holomorphic vector bundles over the supertwistor space $\mathcal{P}^{3|\mathcal{N}}$ which are holomorphically trivial on any $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}}$ and gauge equivalence classes of solutions to the $\mathcal{N}$-extended self-dual SYM equations on $\mathbb{R}^4$. In other words, there is a bijection between the moduli spaces of hCS theory on $\mathcal{P}^{3|\mathcal{N}}$ and the one of self-dual SYM theory on $\mathbb{R}^4$. In fact, our field expansions of $\hat{\mathcal{A}}_\alpha$ and $\hat{\mathcal{A}}_{\bar{\lambda}}$ define the Penrose-Ward transform explicitly.

Signs of Integrability

- Integrable structures in $SU(N)$ $\mathcal{N} = 4$ SYM theory have first been discovered by Minahan and Zarembo in the large N-limit. [J. A. Minahan, K. Zarembo, hep-th/0212208]
- Later on, it has been realized that it's possible to interpret the one-loop dilatation operator as Hamiltonian of an integrable quantum spin chain. [N. Beisert, hep-th/0307015]
- Another development which has pointed towards integrable structures was triggered by Bena, Polchinski and Roiban who showed that the classical Green-Schwarz superstring on $AdS_5 \times S^5$ possesses an infinite number of conserved nonlocal charges.

  [I. Bena, J. Polchinski, R. Roiban, hep-th/0305116]

Signs of Integrability

- Berkovits has then shown that these nonlocal charges also exist after including quantum corrections.
  [N. Berkovits, hep-th/0409159,hep-th/0411170]
- Dolan, Nappi and Witten related these nonlocal charges for the superstring to a corresponding set of nonlocal charges in the gauge theory.
  [L. Dolan, C. Nappi, E. Witten, hep-th/0308089,hep-th/0401243]

Signs of Integrability

- In the following, we'll show how one can (at least classically) construct hidden symmetry algebras (and hence an infinite number of conserved nonlocal charges) in SYM theory via the supertwistor correspondence.

  [M. Wolf, hep-th/0412163]

- For simplicity, let's concentrate on the self-dual subsector of $\mathcal{N} = 4$ SYM theory. However, the presented algorithm also applies (modulo technicalities) to the full theory which is due to the existence of a supertwistor correspondence relating holomorphic vector bundles over a super quadric living inside the superambitwistor space $\mathbb{C}P^{3|3} \times \mathbb{C}P^{3|3}$ and $\mathcal{N} = 4$ SYM theory on $\mathbb{R}^4$.

Some Preliminaries

- Above, we've seen that via the PW-transform we got the bijection:

$$\mathcal{M}_{\rm hol}(\mathcal{P}^{3|\mathcal{N}}) \ \ni\ [f] \ \longleftrightarrow\ [\mathcal{A}] \ \in\ \mathcal{M}^{\mathcal{N}}_{\rm SDYM}$$

- However, we can associate with any open subset $\Omega \subset \mathcal{U}_+ \cap \mathcal{U}_-$ with $\mathcal{P}^{3|\mathcal{N}} = \mathcal{U}_+ \cup \mathcal{U}_-$ an infinite number of such $[f]$.
- Each class $[f]$ corresponds to a class $[\mathcal{A}]$ and vice versa.
  - Q: Can one construct a new solution from a given one?

Some Preliminaries

- In the sequel, we consider infinitesimal deformations of the transition functions and relate them – by virtue of PW – to infinitesimal perturbations of the gauge potential:

$$T_{[f]}\mathcal{M}_{\text{hol}}(\mathcal{P}^{3|\mathcal{N}}) \;\cong\; T_{[\mathcal{A}]}\mathcal{M}^{\mathcal{N}}_{\text{SDYM}}$$

- Remark: The right mathematical tool to describe these deformations is sheaf cohomology.

Infinitesimal Deformations

- Consider $(\mathcal{E}, f_{+-}) \to \mathcal{P}^{3|\mathcal{N}}$. Then a version of Kodaira's theorem tells us that any infinitesimal deformation of f_{+-} is allowed, as small enough perturbations of $\mathcal{E}$ will preserve its trivializability properties on the curves $\mathbb{C}P^1_{x,\eta} \hookrightarrow \mathcal{P}^{3|\mathcal{N}}$.
- Consider now

$$\delta \ : \ f_{+-} \ \mapsto \ \delta f_{+-} \ = \ \sum \epsilon_a \delta_a f_{+-}$$

Thus,

$$f_{+-} + \delta f_{+-} \ = \ (\psi_+ + \delta\psi_+)^{-1}(\psi_- + \delta\psi_-),$$

Infinitesimal Deformations

i.e.,

$$\delta f_{+-} \;=\; f_{+-}\psi_-^{-1}\delta\psi_- - \psi_+^{-1}\delta\psi_+ f_{+-}$$

together with the $\mathfrak{gl}(n,\mathbb{C})$-valued function

$$\varphi_{+-} \;\equiv\; \psi_+(\delta f_{+-})\psi_-^{-1}$$

yields

$$\varphi_{+-} \;=\; \phi_+ - \phi_-, \qquad \delta\psi_\pm \;=\; -\phi_\pm\psi_\pm$$

- The $\phi_\pm$ are holomorphic in $\lambda_\pm$.
- The $\phi_\pm$ are not unique.
- To find $\phi_\pm$ means to solve the infinitesimal Riemann-Hilbert problem.

Infinitesimal Deformations

- From $\mathcal{A}_I^+ = \psi_\pm D_I^+ \psi_\pm^{-1}$, where $I = (\alpha, i)$, we deduce

$$\delta\mathcal{A}_I^+ \;=\; \nabla_I^+\phi_\pm, \qquad \nabla_I^+ \;\equiv D_I^+ + \mathcal{A}_I^+.$$

- Thus, using the twistor functions $\nabla_I^+\phi_\pm$ and performing contour integrals as before, gives the desired deformations $\delta\mathcal{A}_a$, where $a = (\alpha\dot\alpha, i\dot\alpha)$.
- Remark:
  - $\phi_\pm = \psi_\pm\chi_\pm\psi_\pm^{-1} \Longrightarrow \delta\mathcal{A}_a = 0$
  - $\delta\mathcal{A}_a = \nabla_a\omega \Longrightarrow \phi_\pm = \omega \Longrightarrow \delta f_{+-} = 0$

Infinitesimal Deformations

- Thus, we've
$$PW \ : \ [\delta f_{+-}] \ \longleftrightarrow \ [\delta \mathcal{A}_a].$$
- Question:
Let $\{\delta_a\}$ be a set of transformations according to
$\delta_a \ : \ f_{+-} \mapsto f_{+-} + \sum \epsilon_a \delta_a f_{+-}$ satisfying $[\delta_a, \delta_b\} = C_{ab}{}^c \delta_c$.
What is the corresponding symmetry algebra of the e.o.m.?

Example I: Kac-Moody Symmetries

- Let $\mathfrak{g}$ be some Lie superalgebra with

$$[X_a, X_b\} \;=\; f_{ab}{}^c X_c\,.$$

- Define the following perturbation:

$$\delta_a^m f_{+-} \;\equiv\; \lambda_+^m [X_a, f_{+-}], \qquad m \in \mathbb{Z}$$

- Then, it's easy to see that

$$[\delta_a^m, \delta_b^n\} \;=\; f_{ab}{}^c \delta_c^{m+n},$$

i.e., we get a centerless Kac-Moody algebra $\mathfrak{g} \otimes \mathbb{C}[[\lambda, \lambda^{-1}]]$.

Example I: Kac-Moody Symmetries

- Next, we need to find the corresponding algebra on the gauge theory side.
- Algorithm:
  - (i) concrete splitting of $\varphi^m_{+-a} = \psi_+ \delta^m_a f_{+-} \psi_-^{-1}$
  - (ii) action of δ^m_a on $\mathcal{A}_a$
  - (iii) compute $[\delta^m_a, \delta^n_b\}$ on $\mathcal{A}_a$

Example I: Kac-Moody Symmetries

(i) We find

$$\begin{aligned}\varphi^m_{+-a} &= \psi_+\delta^m_a f_{+-}\psi_-^{-1} = \phi^m_{+a} - \phi^m_{-a}\\ &= \psi_+\lambda^m_+[X_a, f_{+-}]\psi_-^{-1} = \cdots = \lambda^m_+\phi^0_{+a} - \lambda^m_+\phi^0_{-a}\end{aligned}$$

together with $\phi^0_{\pm a} = -[X_a, \psi_\pm]\psi_\pm^{-1}$ and

$$\begin{aligned}\phi^m_{+a} &= \sum_{n=0}^{\infty}\lambda_+^{m+n}\phi^{0(n)}_{+a} - \sum_{n=0}^{m-1}\lambda_+^{m-n}\phi^{0(n)}_{-a}\\ \phi^m_{-a} &= \sum_{n=0}^{\infty}\lambda_+^{-n}\phi^{0(m+n)}_{-a}\end{aligned}$$

Example I: Kac-Moody Symmetries

(ii) The transformations of the components of the gauge potential are given by

$$\begin{aligned}\delta^m_a \mathcal{A}_{\alpha\dot{1}} &= \nabla_{\alpha\dot{1}}\phi^{m(0)}_{-a},\\ \delta^m_a \mathcal{A}_{\alpha\dot{2}} &= -\nabla_{\alpha\dot{1}}\phi^{m(1)}_{-a} + \nabla_{\alpha\dot{2}}\phi^{m(0)}_{-a}\end{aligned}$$

and similarly for $\mathcal{A}^i_{\dot{\alpha}}$.

(iii) A lengthy calculation shows that

$$[\delta^m_a, \delta^n_b\} = f_{ab}{}^c\, \delta^{m+n}_c .$$

Thus, we get indeed $\mathfrak{g}\otimes\mathbb{C}[[\lambda,\lambda^{-1}]]$.

Example I: Kac-Moody Symmetries

Some Aside:
We considered the simplest example, namely

$$\delta_a^m f_{+-} = \lambda_+^m [X_a, f_{+-}].$$

But of course, one can take more general transformations, such as

$$\delta f_{+-} = [R, f_{+-}],$$

where R is some arbitrary matrix depending holomorphically on the twistor coordinates. This essentially boils down to the infinitesimal action of the group $C^1(\mathcal{P}^{3|\mathcal{N}}, \mathcal{O}_{Gl})$ on the space $Z^1(\mathcal{P}^{3|\mathcal{N}}, \mathcal{O}_{Gl})$ in Čech terminology.

Example II: Superconformal Symmetries

- Recall that the self-dual SYM equations are conformally invariant. Consider $\mathbb{R}^{4|2\mathcal{N}}$ and let $\{N_a \in \Gamma(T\mathbb{R}^{4|2\mathcal{N}})\}$ be the set of generators of the superconformal group realized as vector fields on $\mathbb{R}^{4|2\mathcal{N}}$. Then

$$\mathcal{A} \ \mapsto \ \mathcal{A} + \delta_N \mathcal{A} \ = \ \mathcal{A} + \mathcal{L}_N \mathcal{A}$$

  gives a symmetry of the e.o.m.
- Remember that the linear system $(D_I^+ + \mathcal{A}_I^+)\psi_\pm = 0$ has the self-dual SYM equations as compatibility condition.

How to define the action of the superconformal group on the supertwistor space $\mathcal{P}^{3|\mathcal{N}}$?

Example II: Superconformal Symmetries

- Remember that the (super)twistor space describes constant almost complex structures on $\mathbb{R}^{4|2\mathcal{N}}$.
- Thus, the action of the superconformal group must preserve a fixed almost complex structure J, i.e.,

$$\mathcal{L}_{\widetilde{N}_a} J = 0.$$

- This condition gives a set of PDEs for the pulled-back vectorfields $\widetilde{N}_a \in \Gamma(T\mathcal{P}^{3|\mathcal{N}})$, which can be solved explicitly.

Example II: Superconformal Symmetries

- Thus,

$$\delta_a \mathcal{A} \;=\; \mathcal{L}_{N_a}\mathcal{A}, \qquad \delta_a \psi_\pm \;=\; \mathcal{L}_{\widetilde{N}_a}\psi_\pm \;=\; \widetilde{N}_a \psi_\pm$$

leaves the linear system together with its compatibility conditions invariant. Furthermore, we've

$$[\delta_a, \delta_b\} \;=\; f_{ab}{}^c \delta_c, \qquad (1)$$

where the $f_{ab}{}^c$ are the structure constants of the superconformal group.

Example II: Superconformal Symmetries

- Next, we consider affine extensions of (1) and define

$$\widetilde{N}_a^m \;\equiv\; \lambda_+^m \widetilde{N}_a^b \partial_b + \lambda_+^m \widetilde{N}_a^{\lambda_+} \partial_{\lambda_+} + \bar{\lambda}_+^m \widetilde{N}_a^{\bar{\lambda}_+} \partial_{\bar{\lambda}_+},$$

  where $m \in \mathbb{Z}$.
  - Note that $\mathcal{L}_{\widetilde{N}_a^m} J = 0$!
- Then, we define

$$\delta_a^m f_{+-} \;\equiv\; \widetilde{N}_a^m f_{+-}.$$

Example II: Superconformal Symmetries

- It's easy to verify that

$$[\delta_a^m, \delta_b^n\} \;=\; ({f_{ab}}^c + n g_a \delta_b^c - (-)^{p_a p_b} m g_b \delta_a^c)\delta_c^{m+n},$$

  with $g_a \equiv \lambda_+^{-1} \widetilde{N}_a^{\lambda_+}$.
- Therefore, depending of what subalgebra of the superconformal algebra we're considering, we obtain pure Kac-Moody and Virasoro algebras – in general though KMV algebras.
- Now, we need to find the corresponding algebra on the gauge theory side. Since it's already late, I won't bother you with details and rather give the results:

Example II: Superconformal Symmetries

(i) The infinitesimal Riemann-Hilbert problem gives:

$$\varphi^m_{+-a} = \psi_+ \delta^m_a \psi_-^{-1} = \lambda^m_+ \phi^0_{+a} - \lambda^m_+ \phi^0_{-a}$$

with $\phi^0_{\pm a} = -(\widetilde{N}_a \psi_\pm)\psi_\pm^{-1}$.

(ii) The transformation laws of the gauge potential are as before, e.g.,

$$\begin{aligned} \delta^m_a \mathcal{A}_{\alpha\dot{1}} &= \nabla_{\alpha\dot{1}} \phi^{m(0)}_{-a}, \\ \delta^m_a \mathcal{A}_{\alpha\dot{2}} &= -\nabla_{\alpha\dot{1}} \phi^{m(1)}_{-a} + \nabla_{\alpha\dot{2}} \phi^{m(0)}_{-a}. \end{aligned}$$

Example II: Superconformal Symmetries

(iii) The algebra reads as

$$[\delta_a^m, \delta_b^n\} \;=\; h_{ab}{}^c\,\delta_c^{m+n} + \sum_k (n g_a^{(k)} \delta_b^c - (-)^{p_a p_b} g_b^{(k)} \delta_a^c)\delta_c^{m+n+k}.$$

Remark:
Here, the $h_{ab}{}^c$ are the structure constants of the maximal subalgebra of the superconformal algebra which doesn't contain $\widetilde{K}^{\alpha\dot{\alpha}}$ and $\widetilde{K}_i^{\dot{\alpha}}$, respectively!

Conclusions

What we have:

- I've explained the supertwistor correspondence relating holomorphic vector bundles over the supertwistor space and self-dual SYM theory in four dimensions.
- I've shown how to construct hidden symmetry algebras of the self-dual SYM equations by using supertwistors.
- In particular, we disussed affine extensions of gauge and spacetime symmetries.

Conclusions

What we have:

There are other aspects/applications which I haven't touched here:

- One can consider certain Abelian subalgebras of the affinely extended superconformal algebra to construct hierarchies of the self-dual SYM equations.
- These lead to infinitely many Abelian symmetries and naturally to enhanced supertwistor spaces.

[M. Wolf, hep-th/0412163]

- They are open subsets of weighted projective spaces, which are in certain situations CY spaces.

Conclusions

What we have:

- The 4D spacetime interpretation of hCS theory on certain weighted projective spaces by virtue of PW has been discussed. [A. D. Popov, M. Wolf, hep-th/0406224]
- The supertwistor correspondence (and the resulting *4D* gauge theories) has been extended to so-called exotic supermanifolds. [C. Sämann, hep-th/0410292]
- One may also consider dimensional reductions and discuss a *3D* version which leads to the mini-supertwistor space, which is CY, and to 3D $\mathcal{N} = 8$ SYM theory. This setup allows also for mass deformations. [D. W. Chiou et al., hep-th/0502076] [to appear]

Outlook

What's next?

- One should generalize the above construction to the full $\mathcal{N} = 4$ SYM theory, which is possible (modulo technicalities) due to the existence of a supertwistor correspondence.
- Hopefully, it will then be possible to discuss quantum corrections to the obtained symmetry algebras in the large *N*-limit using supertwistor techniques.
- ...