Twistor Space, Amplitudes and No-Triangle Hypothesis in Gravity

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Introduction

Twistor space / New insights

Amplitudes N=4,
N=1, QCD
at NLO, Gravity..

Trees

Twistors

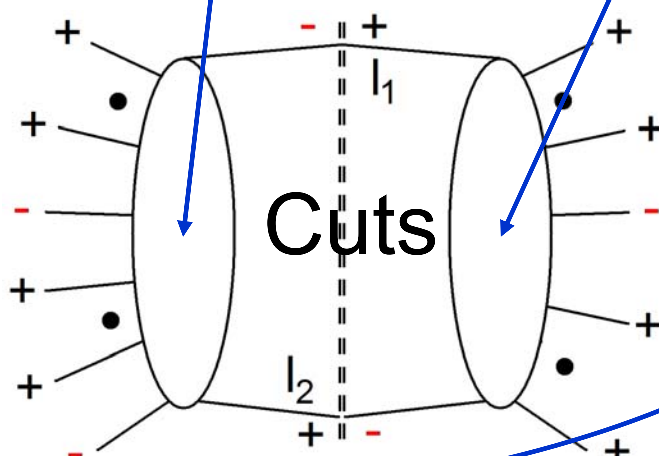
(Witten)

Hidden Beauty!

Loop amplitudes

Simple expressions
for amplitudes

Unitarity



Outline

Quantum gravity from Perturbation Theory

- KLT and spinor helicity formalism
- Twistor space structure

New techniques for gravity tree amplitudes

Loop amplitudes from Unitarity

- N=8 Supergravity
- No-triangle hypothesis
- New insights? .. 2-loops, 3-loops n-loops ..

– Insights from string theory??

Quantum theory for gravity

- Gravity as a theory of point-like interactions
- Non-renormalisable theory!

Dimensionful

$G_N=1/M^2_{planck}$

- Traditional belief : – no known symmetry can remove higher derivative divergences..

String theory can by introducing new length scale

- Focus: N=8 supergravity – maximal supersymmetry

(Cremmer,Julia, Scherk; Cremmer, Julia)

  – Also cancellations in pure gravity as well..

Calculation of perturbative amplitudes

Feynman diagrams:

Factorial Growth!

Momentum vectors :

(p_i ƒp_j)

Generic Feynman amplitudes

Sum over topological different diagrams

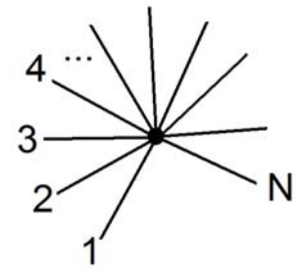
External polarisation tensors :

(p_i ƒε_j) (ε_i ƒε_j)

Amplitudes

Specifying external polarisation tensors $(\varepsilon_i, \varepsilon_j)$

Colour ordering



$\mathrm{Tr}(T_1 T_2 .. T_n)$

Simplifications

Recursion

Spinor-helicity formalism

Loop amplitudes (Unitarity, Supersymmetric decomposition)

Gravity Trees

Gravity Amplitudes

Expand Einstein-Hilbert Lagrangian :

Infinitely many vertices

$$\mathcal{L}_{\mathrm{EH}} = \int d^4x \left[\sqrt{-g}R\right]$$

$$g_{\mu\nu} \equiv \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

Vertices: 3pt, 4pt, 5pt,..n-pt

Feynman diagrams: not attractive...!

Complicated expressions for vertices!

$$\begin{aligned} V^{(3)}_{\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) = \kappa\,\mathrm{sym}\Big[& -\frac{1}{2}P_3(k_1\cdot k_2\,\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \frac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) \\ & + \frac{1}{2}P_3(k_1\cdot k_2\,\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) + P_6(k_1\cdot k_2\,\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) \\ & - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) \\ & + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\,\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})\Big], \end{aligned}$$

45 terms + sym (Sannan)

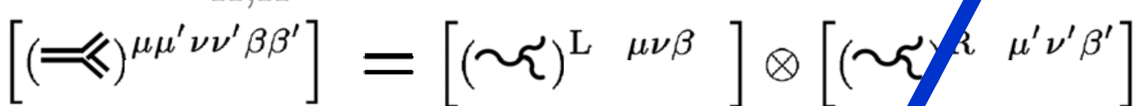
Gravity Amplitudes

KLT relationship (Kawai, Lewellen and Tye)

The KLT relationship relates open and closed strings

Not manifest crossing symmetry

$$A^{M}_{\text{closed}} \sim \sum_{\Pi,\tilde{\Pi}} e^{i\pi\Phi(\Pi,\tilde{\Pi})} A^{\text{left open}}_{M}(\Pi) A^{\text{right open}}_{M}(\tilde{\Pi})$$



KLT not manifestly crossing symmetric – explicit representation :

$$M_3^{\text{tree}}(1,2,3) = -iA_3^{\text{tree}}(1,2,3)A_3^{\text{tree}}(1,2,3)\,,$$

$$M_4^{\text{tree}}(1,2,3,4) = -is_{12}A_4^{\text{tree}}(1,2,3,4)A_4^{\text{tree}}(1,2,4,3)$$

$$M_5^{\text{tree}}(1,2,3,4,5) = is_{12}s_{34}A_5^{\text{tree}}(1,2,3,4,5)A_5^{\text{tree}}(2,1,4,3,5) + is_{13}s_{24}A_5^{\text{tree}}(1,3,2,4,5)A_5^{\text{tree}}(3,1,4,2,5)\,.$$

KLT not the simplest form but better than Feynman diagrams

Momentum prefactors cancel double poles

Simplicity of YM amplitudes!!

Spinor Helicity

Helicity states formalism

Spinor products :

$$\langle i\ j\rangle = \epsilon^{mn}\lambda^i_m\lambda^j_n \quad [i\ j] = \epsilon^{\dot m\dot n}\tilde\lambda^i_{\dot m}\tilde\lambda^j_{\dot n}$$

Different representations of the Lorentz group

$$p_{a\dot a} = \sigma^\mu_{a\dot a}p_\mu$$

$$p^\mu p_\mu = 0 \qquad p_{a\dot a} = \lambda_a\tilde\lambda_{\dot a}$$

Momentum parts of amplitudes:

$$q_{a\dot a}{=}\mu_a\tilde\mu_{\dot a} \quad p_{a\dot a} = \lambda_a\tilde\lambda_{\dot a} \quad 2(p\cdot q) = s_{ij} = -\langle\lambda,\mu\rangle[\tilde\lambda,\tilde\mu]$$

Spin-2 polarisation tensors in terms of helicities, (squares of those of YM):

$$\varepsilon^-_{a\dot a} = \frac{\lambda_a\tilde\mu_{\dot a}}{[\tilde\lambda,\tilde\mu]} \qquad \tilde\varepsilon^+_{a\dot a} = \frac{\mu_a\tilde\lambda_{\dot a}}{\langle\mu,\lambda\rangle} \qquad \begin{matrix}\varepsilon^- & \varepsilon^- \\ \tilde\varepsilon^+ & \tilde\varepsilon^+\end{matrix}$$

(Xu, Zhang, Chang)

Yang-Mills MHV-amplitudes

Tree amplitudes

(n) same helicities vanishes

$$A^{tree}(1^+,2^+,3^+,4^+,..) = 0$$

First non-trivial example,

(M)aximally

(H)elicity (V)iolating

(MHV) amplitudes

(n-1) same helicities vanishes

$$A^{tree}(1^+,2^+,..,j^-,..) = 0$$

(n-2) same helicities:

$$A^{tree}(1^+,2^+,..,j^-,..,k^-,..) \neq 0$$

One single term!!

$A^{tree\ MHV}$ Given by the formula (Parke and Taylor) and proven by (Berends and Giele)

$$i \frac{\langle j\, k\rangle^4}{\langle 1\, 2\rangle \langle 2\, 3\rangle \cdots \langle n\, 1\rangle}$$

Gravity MHV amplitudes

- Can be generated from KLT via YM MHV amplitudes.

$$M_4^{\rm tree}(1^-,2^-,3^+,4^+) = i\,\langle 1\,2\rangle^8\,\frac{[1\,2]}{\langle 3\,4\rangle\;N(4)}$$

$$M_5^{\rm tree}(1^-,2^-,3^+,4^+,5^+) = i\,\langle 1\,2\rangle^8\,\frac{\varepsilon(1,2,3,4)}{N(5)}$$

Anti holomorphic Contributions – feature in gravity

$$\varepsilon(i,j,m,n) \equiv 4i\varepsilon_{\mu\nu\rho\sigma}k_i^\mu k_j^\nu k_m^\rho k_n^\sigma \;=\; [i\,j]\,\langle j\,m\rangle\,[m\,n]\,\langle n\,i\rangle - \langle i\,j\rangle\,[j\,m]\,\langle m\,n\rangle\,[n\,i]$$

- (Berends-Giele-Kuijf) recursion formula

$$\begin{aligned} &M_n^{\rm tree}(1^-,2^-,3^+,\cdots,n^+) \\ &\quad = -i\,\langle 1\,2\rangle^8 \times \Bigg[\frac{[1\,2]\,[n-2\,n-1]}{\langle 1\,n-1\rangle\,N(n)}\Bigg(\prod_{i=1}^{n-3}\prod_{j=i+2}^{n-1}\langle i\,j\rangle\Bigg)\prod_{l=3}^{n-3}\big(-[n|K_{l+1,n-1}|l\rangle\big) \\ &\qquad + \mathcal{P}(2,3,\cdots,n-2)\Bigg] \end{aligned}$$

Simplifications from Spinor-Helicity

$$s_{ij} = -\langle \lambda, \mu \rangle [\tilde{\lambda}, \tilde{\mu}]$$

Huge simplifications

$$\begin{aligned} V^{(3)}_{\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) = \kappa \,\mathrm{sym}\Big[& -\frac{1}{2}P_3(k_1\cdot k_2\,\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \frac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) \\ & + \frac{1}{2}P_3(k_1\cdot k_2\,\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) + P_6(k_1\cdot k_2\,\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) \\ & - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) \\ & + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\,\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})\Big], \end{aligned}$$

45 terms + sym

Vanish in spinor helicity formalism

Contractions

$$\varepsilon^-_{a\dot{a}} = \frac{\lambda_a \tilde{\mu}_{\dot{a}}}{[\tilde{\lambda}, \tilde{\mu}]} \qquad \tilde{\varepsilon}^+_{a\dot{a}} = \frac{\mu_a \tilde{\lambda}_{\dot{a}}}{\langle \mu, \lambda \rangle}$$

Gravity:

$\varepsilon^- \; \varepsilon^-$

$\tilde{\varepsilon}^+ \; \tilde{\varepsilon}^+$

$$A_3(1^-, 2^-, 3^+) = -i\frac{\langle 1\,2\rangle^6}{\langle 2\,3\rangle \langle 3\,1\rangle}$$

Scattering amplitudes in D=4

Amplitudes in gravity theories as well as YM can hence be expressed completely specifying

The external helicities

e.g. : $A(1^+,2^-,3^+,4^+, ..)$

The spinor variables

$$\lambda_i \quad \bar{\lambda}_i$$

Spinor Helicity formalism

Note on notation

We will use the notation:

$$s_{i,i+1} \equiv K_{i,i+1}^2 = (p_i + p_{i+1})^2 \qquad \text{and} \qquad t_{i,j} \equiv K_{i,j}^2 = (p_i + \ldots + p_j)^2$$

$$\langle i\ j\rangle = \epsilon^{mn}\lambda_m^i\lambda_n^j \qquad [i\ j] = \epsilon^{\dot{m}\dot{n}}\tilde{\lambda}_{\dot{m}}^i\tilde{\lambda}_{\dot{n}}^j$$

$$[k|K_{i,j}|l\rangle \equiv \langle k^+|\not{K}_{i,j}|l^+\rangle \equiv \langle l^-|\not{K}_{i,j}|k^-\rangle \equiv \langle l|K_{i,j}|k] \equiv \sum_{a=i}^{j} [k\,a]\langle a\,l\rangle$$

$$\langle k|K_{i,j}K_{m,n}|l\rangle \equiv \langle k^-|\not{K}_{i,j}\not{K}_{m,n}|l^+\rangle \equiv \sum_{a=i}^{j}\sum_{b=m}^{n} \langle k\,a\rangle\,[a\,b]\,\langle b\,l\rangle$$

Traces...

$$2(p\cdot q) = s_{ij} = -\langle\lambda,\mu\rangle[\tilde{\lambda},\tilde{\mu}]$$

Twistor space Properties

Twistor space

- Transformation of amplitudes into twistor space (Penrose)

$$\tilde{\lambda}_{\dot{a}} \rightarrow i\frac{\partial}{\partial \mu^{\dot{a}}}, \qquad -i\frac{\partial}{\partial \tilde{\lambda}_{\dot{a}}} \rightarrow \mu_{\dot{a}}$$

- In metric signature (+ + - -) : 2D Fourier transform

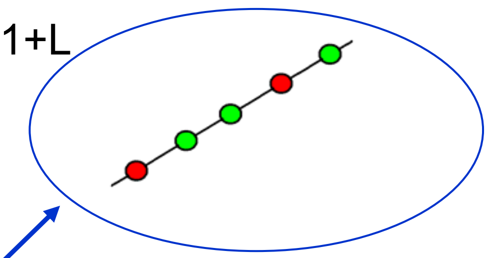
$$\Phi(\mu) = \int \frac{d^2\tilde{\lambda}}{(2\pi)^2} \exp(i\mu^{\dot{a}}\tilde{\lambda}_{\dot{a}})\Phi(\tilde{\lambda})$$

- In twistor space : plane wave function is a line:

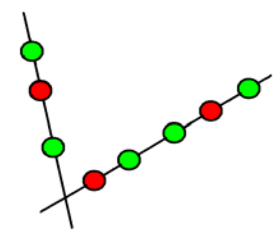
$$\int \frac{d^2\tilde{\lambda}}{(2\pi)^2} \exp(i\mu^{\dot{b}}\tilde{\lambda}_{\dot{b}}) \exp(ix^{a\dot{a}}\lambda_a\tilde{\lambda}_{\dot{a}}) = \delta^2(\mu_{\dot{a}} + x_{a\dot{a}}\lambda^a)$$

- Tree amplitudes in YM on degenerate algebraic curves

Degree : N-1+L



(Witten)



Degree : number of negative helicities

Collinear and Coplanar Operators

Another way to look at twistor space support..

$$[F_{ijk}, \eta] = \langle i\, j\rangle \left[\frac{\partial}{\partial \tilde{\lambda}_k}, \eta\right] + \langle j\, k\rangle \left[\frac{\partial}{\partial \tilde{\lambda}_i}, \eta\right] + \langle k\, i\rangle \left[\frac{\partial}{\partial \tilde{\lambda}_j}, \eta\right]$$

$$K_{ijkl} = \frac{1}{4}\Big[\langle ij\rangle \epsilon^{\dot{a}\dot{b}} \frac{\partial}{\partial \tilde{\lambda}_k^{\dot{a}}} \frac{\partial}{\partial \tilde{\lambda}_l^{\dot{b}}} - \langle ik\rangle \epsilon^{\dot{a}\dot{b}} \frac{\partial}{\partial \tilde{\lambda}_j^{\dot{a}}} \frac{\partial}{\partial \tilde{\lambda}_l^{\dot{b}}} + \langle il\rangle \epsilon^{\dot{a}\dot{b}} \frac{\partial}{\partial \tilde{\lambda}_j^{\dot{a}}} \frac{\partial}{\partial \tilde{\lambda}_k^{\dot{b}}} + \langle jk\rangle \epsilon^{\dot{a}\dot{b}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{a}}} \frac{\partial}{\partial \tilde{\lambda}_l^{\dot{b}}} + \langle jl\rangle \epsilon^{\dot{a}\dot{b}} \frac{\partial}{\partial \tilde{\lambda}_k^{\dot{a}}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{b}}} - \langle kl\rangle \epsilon^{\dot{a}\dot{b}} \frac{\partial}{\partial \tilde{\lambda}_j^{\dot{a}}} \frac{\partial}{\partial \tilde{\lambda}_i^{\dot{b}}}\Big]$$

$$F_{ijk} A_n^{\text{tree MHV}}(1.....n) = 0$$

$$K_{ijkl} A_n^{\text{tree NMHV}}(1.....n) = 0$$

Review: CSW expansion of YM amplitudes

- In the CSW-construction : off-shell MHV-amplitudes building blocks for more complicated amplitude expressions (Cachazo, Svrcek and Witten)
- MHV vertices:

Vertex construction $

spin off of twistor support properties

$$V(1,2,\ldots,r^-,\ldots,s^-,\ldots,n-1,Q^+) = \frac{\langle\lambda_r,\lambda_s\rangle^4}{\prod_{i=1}^{n-1}\langle\lambda_i,\lambda_{i+1}\rangle\langle\lambda_{n-1},\lambda_Q\rangle\langle\lambda_Q,\lambda_1\rangle}$$

$$V(1,2,\ldots,r^-,\ldots,n-1,Q^-) = \frac{\langle\lambda_r,\lambda_Q\rangle^4}{\prod_{i=1}^{n-1}\langle\lambda_i,\lambda_{i+1}\rangle\langle\lambda_{n-1},\lambda_Q\rangle\langle\lambda_Q,\lambda_1\rangle}$$

$$\langle\lambda_i,\lambda_Q\rangle = \sum_j\langle\tilde\lambda_i,\tilde\lambda_{q_j}\rangle[\lambda_{q_j},\eta\rangle = \langle i|Q|\eta]$$

Twistor space properties for gravity

- Twistor-space properties N=8 Supergravity:

  More complicated!

$$A_n^{\text{tree MHV}}(1^+, 2^+, \ldots, p^-, \ldots, q^-, \ldots, n^+)(\lambda_i, \mu_i) \sim$$

N=4

$$\sim \int d^4x \prod_{i=1}^{n} \delta^2(\mu_{i\dot{a}} + x_{a\dot{a}}\lambda_i^a) A_n^{\text{tree MHV}}(1^+, 2^+, \ldots, p^-, \ldots, q^-, \ldots, n^+)(\lambda_i)$$

δ–functions

Signature of non-locality $ typical in gravity

Anti-holomorphic pieces in gravity amplitudes

Derivatives of δ- functions

N=8

$$M_n^{\text{tree MHV}}(1^+, 2^+, \ldots, p^-, \ldots, q^-, \ldots, n^+)(\lambda_i, \mu_i) \sim \int d^4x P\left(-i\frac{\partial}{\partial \mu_{i\dot{a}}}\right) \prod_{i=1}^{n} \delta^2(\mu_{i\dot{a}} + x_{a\dot{a}}\lambda_i^a)$$

Twistor space properties

- For gravity : Guaranteed that

Acting with differential operators F and K

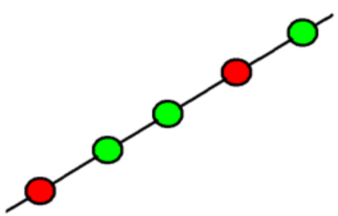
$$F^P M_n^{\text{tree MHV}}(1\ldots n) = 0\,, \qquad \text{for } P > 2(n-3)$$

- Five-point amplitude. (Giombi, Ricci, Rables-Llana and Trancanelli; Bern, NEJBB and Dunbar)

$$K^2 M_5^{\text{tree googly}} = K K' M_5^{\text{tree googly}} = 0$$

- Tree amplitudes :

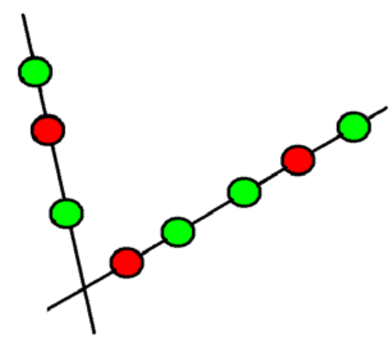
Gravity



$$F^4_{ijk} M_6^{\text{tree MHV}} = 0$$
$$F^5_{ijk} M_7^{\text{tree MHV}} = 0$$
$$F^6_{ijk} M_8^{\text{tree MHV}} = 0$$

$$F^{n-2}_{ijk} M_n^{\text{tree MHV}} = 0$$
$$K^{n-3}_{ijkl} M_n^{\text{tree NMHV}} = 0$$

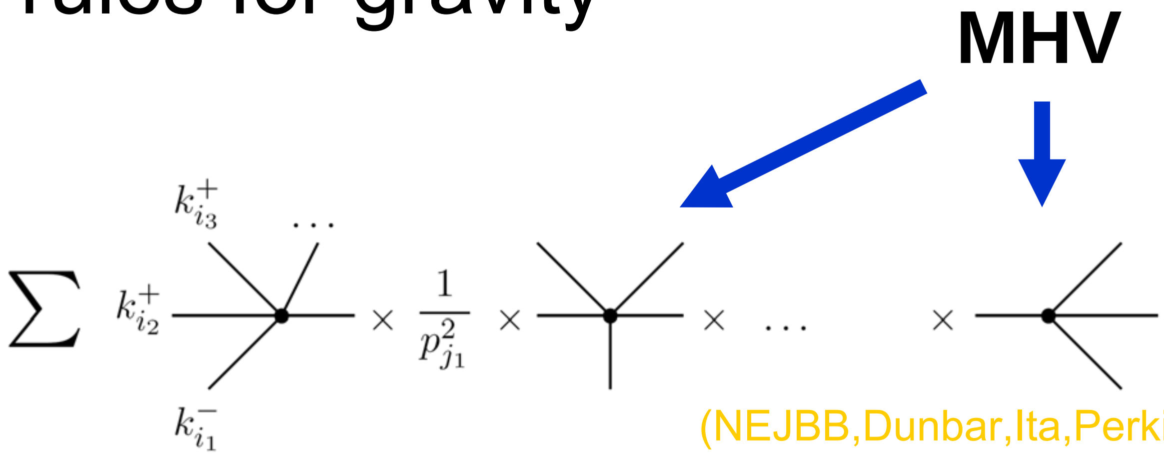
$$K^3_{ijkl} M_6^{\text{tree}\,(---+++)} = 0$$
$$K^4_{ijkl} M_7^{\text{tree}\,(---++++)} = 0$$



(Bern, NEJBB and Dunbar)

Gravity tree properties

MHV rules for gravity



(NEJBB,Dunbar,Ita,Perkins,Risager)

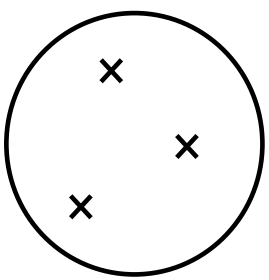
Recursion

$$A(0) = -\sum_\alpha \mathrm{Res}_{z=z_\alpha} \frac{A(z)}{z}$$

(Bedford, Brandhuber,Spence, Travaglini; Cachazo, Svrtec; NEJBB, Dunbar, Ita)

$$\tilde\lambda_a \to \tilde\lambda_a + z\tilde\lambda_b$$

$$\lambda_b \to \lambda_b - z\lambda_a$$

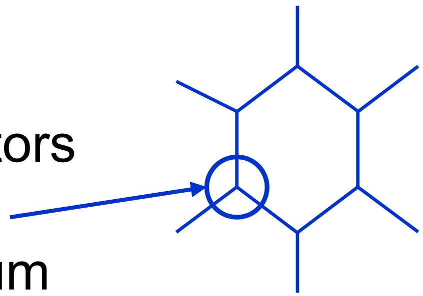


Gravity loops

General 1-loop amplitudes

n-pt amplitude

Vertices carry factors of loop momentum



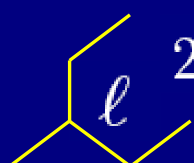
$$\int d^4\ell \frac{F^p(\ell, k, \epsilon)}{\Pi_i p_i^2}$$

p = 2n for gravity

p=n for YM

Propagators

(Passarino-Veltman) reduction

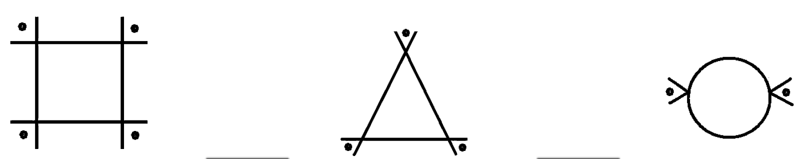


$$2(k\cdot\ell) = (k-\ell)^2 - \ell^2$$

Collapse of a propagator

$$I_r[P^m(l)] \longrightarrow \sum_i I^i_{r-1}[P^{m-1}(l)]$$

$$I^i_4[P^{m'}(l)] \longrightarrow c_i\, I^i_4[1] + \sum_j I^j_3[P^{m'-1}(l)]$$

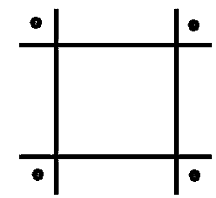


$$M^{1-\mathrm{loop}} = \sum_a c_a I^a_4 + \sum_a d_a I^a_3 + \sum_a e_a I^a_2 + R$$

No-Triangle Hypothesis

Justified suggestion…….

$$M^{\text{1-loop}}_{\mathcal{N}=8} = \sum_{i \in \mathcal{C}} c_i \, I_4^i$$



Consequence: N=8 supergravity same one-loop structure as N=4 SYM

Evidence?

	True for 4pt	(Green,Schwarz,Brink)
	n-point MHV	(Bern,Dixon,Perelstein,Rozowsky)
	6pt NMHV (IR)	(Bern, NEJBB, Dunbar,Ita)
Direct evaluation of cuts	6pt Proof	(NEJBB, Dunbar,Ita, Perkins, Risager)
	7pt evidence	

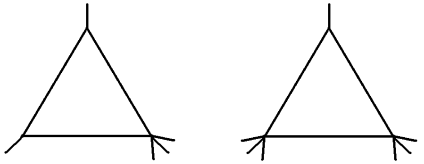
Factorisation suggests this is true for all one-loop amplitudes

No-Triangle Hypothesis by Cuts

Attack different parts of amplitudes 1) .. 2) .. 3) ..

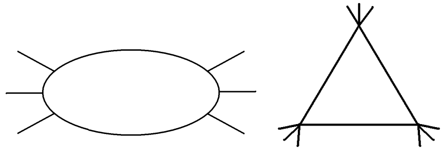
(1) Look at soft divergences (IR)

$ 1m and 2m triangles



(2) Explicit unitary cuts

$ bubble and 3m triangles



(3) Factorisation

$ rational terms.

(NEJBB, Dunbar,Ita, Perkins, Risager)

Infrared for loops (1)

Gravity IR loop relation :

$$M^{\text{one-loop}}_{\epsilon^{-1}}(1,2,\ldots,n) = i c_{\Gamma} \kappa^2 \times \left(\frac{\sum_{i<j} s_{ij} \ln(-s_{ij})}{2\epsilon} \right) \times M^{\text{tree}}(1,2,\ldots,n)$$

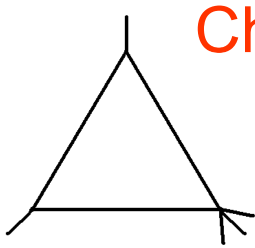
Compact result for SYM tree amplitudes (Bern, Dixon and Kosower; Roiban Spradlin and Volovich)

Check that boxes gives the correct IR divergences

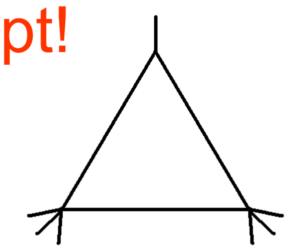
No one mass and two mass triangles

(no statement about three mass triangles

Checked until 7pt!



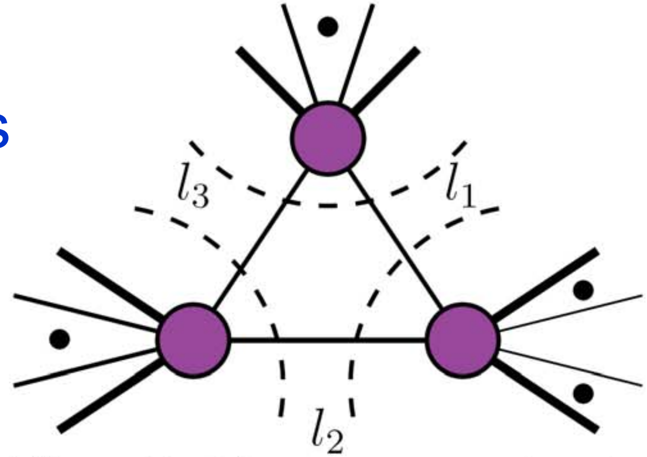
x C(1m) = 0

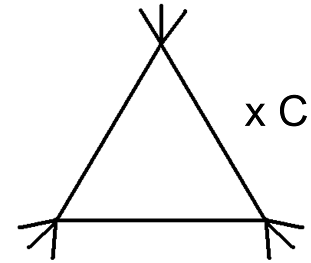


x C(2m) = 0

No-Triangle Hypothesis (2)

Three mass triangles





x C(3m) = 0

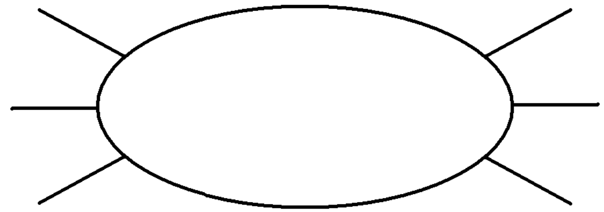
$$C_3 = \sum_{h_i \in \mathcal{S}'} \int d^4 l_1 \delta(l_1^2)\delta(l_2^2)\delta(l_3^2) M\big((l_1)^{h_1}, i_m, \cdots i_j, (-l_2)^{-h_2}\big)$$
$$\times M\big((l_2)^{h_2}, i_{j+1}, \cdots i_l, (-l_3)^{-h_3}\big) \times M\big((l_3)^{h_3}, i_{l+1}, \cdots i_{m-1}, (-l_1)^{-h_1}\big)$$

$$C_3 = \sum c_i (I_4^i)_{\text{triple-cut}} + d_{3m} (I_3^{3m})_{\text{triple-cut}}$$

$$d_3^{3m}[\{1^-, 2^-\}, \{3^-, 4^+\}, \{5^+, 6^+\}] = 0$$
$$d_3^{3m}[\{1^-, 4^+\}, \{2^-, 5^+\}, \{3^-, 6^+\}] = 0$$
$$d_3^{3m}[\{1^-, 2^-\}, \{3^-, 4^+\}, \{5^+, 6^+, 7^+\}] = 0$$
$$d_3^{3m}[\{1^-, 2^-\}, \{3^-, 4^+, 5^+\}, \{6^+, 7^+\}] = 0$$
$$d_3^{3m}[\{1^-, 2^-, 4^+\}, \{3^-, 5^+\}, \{6^+, 7^+\}] = 0$$
$$d_3^{3m}[\{1^-, 4^+\}, \{2^-, 5^+\}, \{3^-, 6^+, 7^+\}] = 0$$

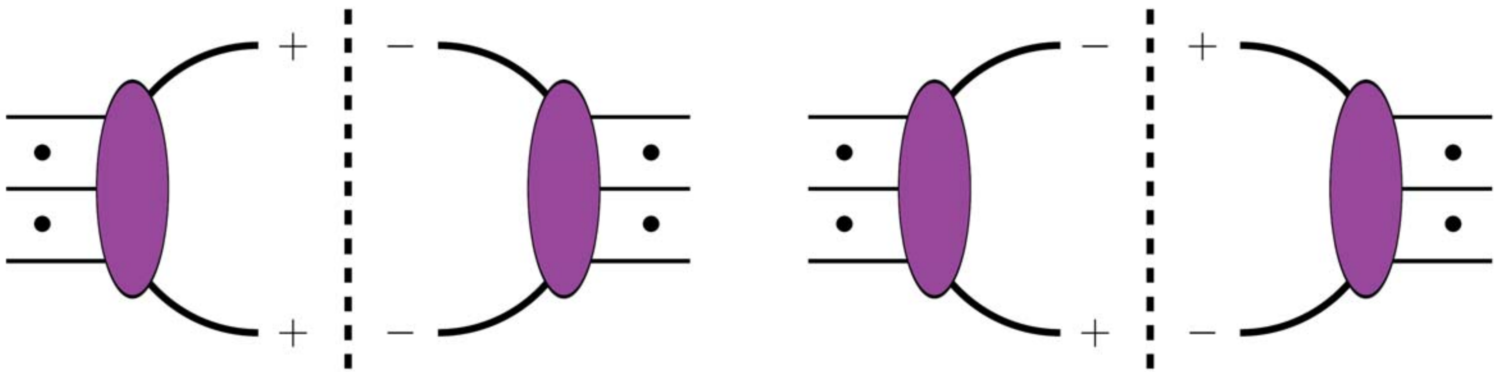
No-Triangle Hypothesis (2)

Evaluate double cuts
Directly using various methods,
Identify singularities.



x C(bubble) = 0

(e.g. Buchbinder, Britto,Cachazo Feng,Mastrolia)



SINGLET **NON-SINGLET**

$$C_{12} = i \int d\mu \, M_4^{\text{tree}}\left((-l_1)^+, 1^-, 2^-, (l_2)^+\right) \times M_6^{\text{tree}}\left((l_1)^-, 3^-, 4^+, 5^+, 6^+, (-l_2)^-\right)$$

where $d\mu = d^4 l_1 d^4 l_2 \delta^{(+)}(l_1^2) \delta^{(+)}(l_2^2) \delta^{(4)}(l_1 - l_2 - k_1 - k_2)$,

No-Triangle Hypothesis (3)

No possibility for rational pieces until 7pt..

Bootstrap methods for QCD should work similarly for the rational parts of N=8

(Berger, Bern, Dixon,Forde, Kosower; Su, Xiao, Yang, Zhu)

Multiparticle factorisation and other physical limits such as soft, collinear makes huge constrains on the possibilities of having a rational term at (n)pt

No-triangle for multiloops

- No-triangle hypothesis 1-loop
  - Consequences for powercounting arguments above one-loop..

$$D < 10/L + 2$$

Two-particle cut might miss certain cancellations

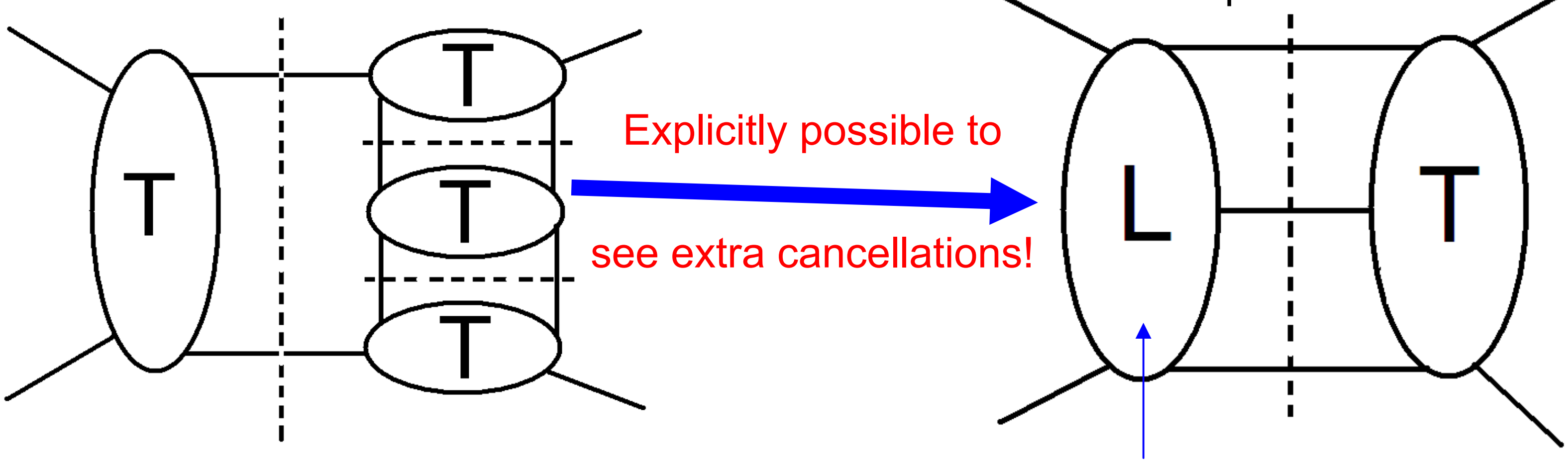
Possible to obtain YM bound??

$D < 6/L + 4$ for gravity???

Bound might be too conservative!!

Iterated two-particle cut

Three/N-particle cut



Three-Loop SYM/ Supergravity

- Three-loop four-point amplitude of N=8 supergravity directly constructed via unitarity.

- The amplitude is ultraviolet finite in four dimensions.

- Degree of divergence in D dimensions at three loop to be no worse than that of N=4 super-Yang-Mills theory. Confirms ‘no-triangle hypothesis’ for three loops.

  – Remark: Surprising extra cancellations between diagrams which are not just ‘triangle-type’..

(Bern, Carrasco, Dixon, Johansson, Kosower, Roiban)

N=8/N=4 UV pattern

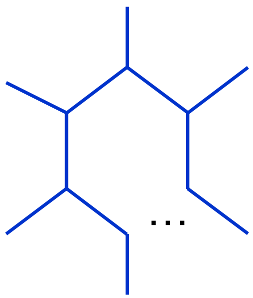
Honest calculation/ conjecture

D=11	0	#/ε				
D=10	0(!)	#/ε				
D=9	0	#/ε				
D=8	#/ε	$\#'/\varepsilon^{2} + \#''/\varepsilon$				
D=7		#/ε				
D=6	0	0				
D=5	0	0	0			
D=4	0	0	0	0		
	L=1	L=2	L=3	L=4	L=5	L=6

N=8 SUGRA

N=4 SYM

Explicit cancellations

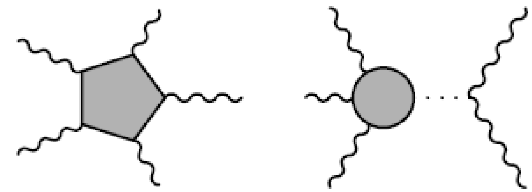


String theory limit of n-pt function

$\alpha^{3}\ \$\ \ 0$

$\tau_2\ \$\ \ 0$

Field theory limit of IIA and IIB string theory on a torus



(a) (b)

Two contributions

Limit finite dist between operators (a) Irred

Pinch interactions (colliding vertices) (b) Red

$$\mathcal{M}_5^{1PI} \;=\; C_5^{(D)}\, I_5^{(D)}\left[|\mathcal{A}_5^{(1)\infty}|^2\right] + \pi^{-1} C_5^{(D+2)}\, I_5^{(D+2)}\left[\mathcal{A}_5^{(2)\infty}\right] \qquad \text{(a)}$$

$$\mathcal{M}_5^{1PR} \;=\; \lim_{\alpha'\to 0} \kappa_{(D)}^{-2} \mathcal{A}_{5,1PR}^{e/e} = \pi^{\frac{D-8}{2}} \Gamma\left(\frac{8-D}{2}\right) \sum_{i\neq j} t_{(ij)} \prod_{r=1}^{4} \int_0^1 d\nu_r\, Q_4(P_i)^{\frac{D}{2}-4}\, \delta(\nu_4 - 1) \qquad \text{(b)}$$

$$I_n^{(D)}[f(\nu)] \;\equiv\; \prod_{i=1}^{n} \int_0^1 d\nu_i\, f(\nu_i)\, Q_n^{\frac{D}{2}-n}\, \delta(\nu_n - 1)$$

$$Q_n(k_i) \;=\; \sum_{1\leq i<j\leq n} (k_i\cdot k_j)\left[(\nu_i-\nu_j)^2 - |\nu_i-\nu_j|\right]$$

(NEJBB and P. Vanhove)
hep-th/0802.0868

Explicit cancellations

$$\mathcal{A}_5^{(1)\infty} = t_{10}\cdot F^5 + \pi \sum_{i\neq j}(h_i\cdot k_j)\dot{G}_B(\nu_i-\nu_j)\,(t_8\cdot F_{\hat{i}}^4)$$
$$= t_{10}\cdot F^5 - \frac{\pi}{2}\sum_{i\neq j}(h_i\cdot k_j)\,G_F(\nu_i-\nu_j)\,(t_8\cdot F_{\hat{i}}^4) - \pi\, H\cdot K_{[5]}$$
$$\mathcal{A}_5^{(2)\infty} = \sum_{i\neq j} h_i\cdot \bar{h}_j\,(t_8\cdot F_{\hat{i}}^4)(t_8\cdot F_{\hat{j}}^4)$$

$$H = \sum_{i=1}^{5} h_i\,(t_8\cdot F_{\hat{i}}^4)$$
$$\bar{H} = \sum_{i=1}^{5} \bar{h}_i\,(t_8\cdot \bar{F}_{\hat{i}}^4)$$

$$\mathcal{M}_5^{1PI} = C_5^{(D)}\, I_5^{(D)}\left[|\mathcal{A}_5^{(1)\infty}|^2\right] + \pi^{-1} C_5^{(D+2)}\, I_5^{(D+2)}\left[\mathcal{A}_5^{(2)\infty}\right]$$

$$M_5[1] = C_5^{(D)} I_5^{(D)}\left[\left|t_{10}\cdot F^5 - \frac{\pi}{2}\sum_{i\neq j}(h_i\cdot k_j)G_F(\nu_i-\nu_j)\,(t_8\cdot F_{\hat{i}}^4)\right|^2\right]$$
^ $\sum$ 1m Boxes

$$M_5[\nu] = -\pi\, C_5^{(D)} I_5^{(D)}\left[\left(t_{10}\cdot F^5 - \frac{\pi}{2}\sum_{i\neq j}(k_i\cdot h_j)G_F(\nu_i-\nu_j)\,(t_8\cdot F_{\hat{i}}^4)\right)(H\cdot K_{[5]})\right]$$
^ $\sum$ 1m Boxes

$$M_5[\nu^2] = \pi^2\, C_5^{(D)}\,(t_8\cdot F_{\hat{i}}^4)(t_8\cdot F_{\hat{j}}^4)\, I_5^{(D)}\left[\,(H\cdot K_{[5]})\,(\bar{H}\cdot K_{[5]})\right]$$
^ $\sum$ 1m Boxes + $\sum$ Triangles

Potential dangerous terms

Explicit cancellations

$$\mathcal{R}^{1PI} = -I_5^{(D)}\Big[(k_4\cdot K_{[5]})(k_5\cdot K_{[5]})\Big]$$
$$-\frac{1}{2}\,I_5^{(D)}\Big[(\sum_{i=1}^{5}(k_5\cdot k_i)\,\mathrm{sign}(\nu_5-\nu_i))(k_4\cdot K_{[5]})\Big]+(4\leftrightarrow 5)$$
$$-\frac{1}{4}\,I_5^{(D)}\Big[\sum_{i,j=1}^{5}(k_5\cdot k_i)\,\mathrm{sign}(\nu_5-\nu_i)(k_4\cdot k_j)\,\mathrm{sign}(\nu_4-\nu_j)\Big]$$
$$-(k_4\cdot k_5)\,I_5^{(D+2)}[1]$$

Cancellations of longitudinal modes

$\hat{}\ \ \varepsilon_i\ \$\ \ k_i$

$$\mathcal{R}^{1PR}\equiv\lim_{\alpha'\to 0}\lim_{4\to 5}\mathcal{R} = -(k_4\cdot k_5)\,I_4^{(45)}[1]$$

Integrals are given as:

$$I_n^{(D)}[f(\nu)]\equiv\prod_{i=1}^{n}\int_0^1 d\nu_i\,f(\nu_i)\,Q_n^{\frac{D}{2}-n}\,\delta(\nu_n-1)$$

$$I_5^{(D)}[(k_4\cdot K_{[5]})(k_5\cdot K_{[5]})] = (k_4\cdot k_5)\,I_5^{(D+2)}[1]$$
$$+\frac{1}{2}\,I_5^{(D)}\Big[(\sum_{i=1}^{5}(k_5\cdot k_i)\,\mathrm{sign}(\nu_5-\nu_i))(k_4\cdot K_{[5]})\Big]+(4\leftrightarrow 5)$$
$$+\frac{1}{4}\,I_5^{(D)}\Big[\sum_{i,j=1}^{5}(k_5\cdot k_i)\,\mathrm{sign}(\nu_5-\nu_i)(k_4\cdot k_j)\,\mathrm{sign}(\nu_4-\nu_j)\Big]$$
$$+(k_4\cdot k_5)\,I_4^{(45)}[1]$$

Sum over orderings

Exact Identity

Explicit cancellations

Now cancellations

$$H = \sum_{i=1}^{4} c_i\, k_i + q_\perp, \qquad \bar{H} = \sum_{i=1}^{4} \bar{c}_i\, k_i + \bar{q}_\perp$$

Express H and bar H
In basis of momentum vector

Only needed
beyond D = 4

$$M_5[\nu^2] = \pi^2\, I_5^{(D)} \Big[\,(H \cdot K_{[5]})\, (\bar{H} \cdot K_{[5]}) \Big]$$

$$\sum_{i=1}^{6} h_i t_i = \begin{cases} \sum_{i=1}^{4} c_i\, k_i & \text{for } D = 4 \\ \sum_{i=1}^{5} c_i\, k_i + q_\perp & \text{for } D \geq 5 \end{cases}$$

Irred part of amplitude

5pt

Cancellation of triangles!!

$$M_5[\nu^2] \propto \sum_{i,j=1}^{4} c_i \bar{c}_j\, I_5^{(D)} \Big[(k_i \cdot K_{[5]})\, (k_j \cdot K_{[5]}) \Big]$$

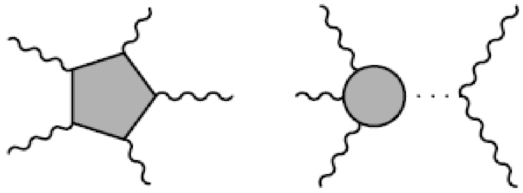
6pt

Conclusion: no triangles from Irred part of amplitude by cancellations of longitudinal modes
no triangles from Red part of amplitude by supersymmetry

Explainations + things to do

No-triangles

- String based rules
  - Field theory limit of string theory used to generate results.



Hint:
- Gauge symmetry

Longitudinal modes

+ crossing symmetry !
Link to cancellations at 1-loop and trees

(Bern, Carrasco, Forde, Ita, Johansson and Arkani-Hamed and Kaplan)

- Further investigations no-triangle hypothesis
  - 5pt ! 6pt and higher. (String based rules useful)
  - Gauge symmetry cancellations at multi-loop level

Conclusions

– More perturbative calculations of loop amplitudes
$ helpful to understand cancellations…

– Will theories with less supersymmetry have similar surprising cancellations??

According to string based analysis most of cancellations are in the Irred part of amplitude

– KLT : Gravity ~ (Yang Mills) x (Yang Mills')
seems to play some role (even at loop level)
although not critical for observed cancellations

Conclusions

- The calculation of gravity amplitudes benefit hugely from the use of new techniques developed for gauge theories.

- Both recursion and MHV –vertex formulations for the calculation of gravity amplitudes exist.

- The perturbative expansion of N=8 seems to be surprisingly simple and very similar to N=4 at one-loop. At three loop no worse UV-divergences than N=4!

- This may have important consequences .. (Berkovits)
  - Hints from String theory?? Explaination ??? (Green, Russo, Vanhove)

  - Perturbative finite / Non-perturbative completion??? (Abou-Zeid, Hull and Mason)

Twistor-string theory for gravity?? – likely if perturbative finiteness holds

Mass-less modes with non-perturbative origin?? (Schnitzer)

(Green, Ooguri, Schwarz)