2 Signal Processing Fundamentals

We can't hope to cover all the important details of one- and two-dimensional signal processing in one chapter. For those who have already seen this material, we hope this chapter will serve as a refresher. For those readers who haven't had prior exposure to signal and image processing, we hope that this chapter will provide enough of an introduction so that the rest of the book will make sense.

All readers are referred to a number of excellent textbooks that cover one- and two-dimensional signal processing in more detail. For information on 1-D processing the reader is referred to [McG74], [Sch75], [Opp75], [Rab75]. The theory and practice of image processing have been described in [Ros82], [Gon77], [Pra78]. The more general case of multidimensional signal processing has been described in [Dud84].

2.1 One-Dimensional Signal Processing

2.1.1 Continuous and Discrete One-Dimensional Functions

One-dimensional continuous functions, such as in Fig. 2.1(a), will be represented in this book by the notation

$$x(t) \tag{1}$$

where $x(t)$ denotes the value as a function at t. This function may be given a discrete representation by sampling its value over a set of points as illustrated in Fig. 2.1(b). Thus the discrete representation can be expressed as the list

$$\cdots\ x(-\tau),\ x(0),\ x(\tau),\ x(2\tau),\ \cdots,\ x(n\tau),\ \cdots\ . \tag{2}$$

As an example of this, the discrete representation of the data in Fig. 2.1(c) is

$$1,\ 3,\ 4,\ 5,\ 4,\ 3,\ 1. \tag{3}$$

It is also possible to represent the samples as a single vector in a multidimensional space. For example, the set of seven samples could also be represented as a vector in a 7-dimensional space, with the first element of the vector equal to 1, the second equal to 3, and so on.

There is a special function that is often useful for explaining operations on functions. It is called the Dirac delta or impulse function. It can't be defined

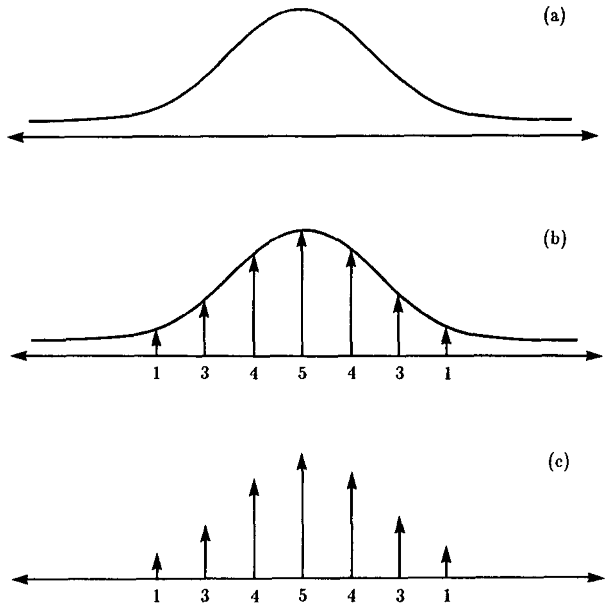


Fig. 2.1: *A one-dimensional signal is shown in (a) with its sampled version in (b). The discrete version of the signal is illustrated in (c).*

directly; instead it must be expressed as the limit of a sequence of functions. First we define a new function called rect (short for rectangle) as follows

$$\text{rect }(t)=\begin{cases}1 & |t|<\frac{1}{2}\\ 0 & \text{elsewhere.}\end{cases} \tag{4}$$

This is illustrated in Fig. 2.2(a). Consider a sequence of functions of ever decreasing support on the t-axis as described by

$$\delta_n(t)=n\ \text{rect }(nt) \tag{5}$$

and illustrated in Fig. 2.2(b). Each function in this sequence has the same area but is of ever increasing height, which tends to infinity as $n \to \infty$. The limit of this sequence of functions is of infinite height but zero width in such a manner that the area is still unity. This limit is often pictorially represented as shown in Fig. 2.2(c) and denoted by $\delta(t)$. Our explanation leads to the definition of the Dirac delta function that follows

$$\int_{-\infty}^{\infty} \delta(t)\ dt=1. \tag{6}$$

The delta function has the following "sampling" property

$$\int_{-\infty}^{\infty} x(t)\delta(t-t')\ dt=x(t') \tag{7}$$

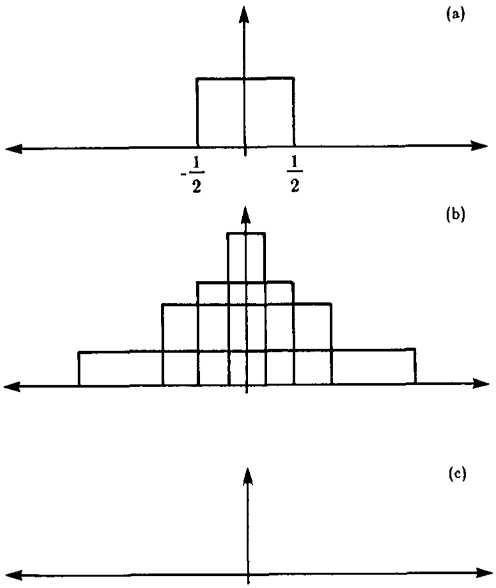


Fig. 2.2: *A rectangle function as shown in (a) is scaled in both width and height (b). In the limit the result is the delta function illustrated in (c).*

where $\delta(t - t')$ is an impulse shifted to the location $t = t'$. When an impulse enters into a product with an arbitrary $x(t)$, all the values of $x(t)$ outside the location $t = t'$ are disregarded. Then by the integral property of the delta function we obtain (7); so we can say that $\delta(t - t')$ samples the function $x(t)$ at t'.

2.1.2 Linear Operations

Functions may be operated on for purposes such as filtering, smoothing, etc. The application of an operator O to a function $x(t)$ will be denoted by

$$O[x(t)]. \tag{8}$$

The operator is linear provided

$$O[\alpha x(t)+\beta y(t)]=\alpha O[x(t)]+\beta O[y(t)] \tag{9}$$

for any pair of constants α and β and for any pair of functions $x(t)$ and $y(t)$.

An interesting class of linear operations is defined by the following integral form

$$z(t)=\int_{-\infty}^{\infty} x(t')h(t,\ t')\ dt' \tag{10}$$

where h is called the impulse response. It is easily shown that h is the system response of the operator applied to a delta function. Assume that the input function is an impulse at $t = t_0$ or

$$x(t)=\delta(t-t_0). \tag{11}$$

Substituting into (10), we obtain

$$z(t)=\int_{-\infty}^{\infty} \delta(t'-t_0)h(t,\ t')\ dt' \tag{12}$$

$$=h(t,\ t_0). \tag{13}$$

Therefore $h(t,\ t')$ can be called the impulse response for the impulse applied at t'.

A linear operation is called shift invariant when

$$y(t)=O[x(t)] \tag{14}$$

implies

$$y(t-\tau)=O[x(t-\tau)] \tag{15}$$

or equivalently

$$h(t,\ t')=h(t-t'). \tag{16}$$

This implies that when the impulse is shifted by t', so is the response, as is further illustrated in Fig. 2.3. In other words, the response produced by the linear operation does not vary with the location of the impulse; it is merely shifted by the same amount.

For shift invariant operations, the integral form in (10) becomes

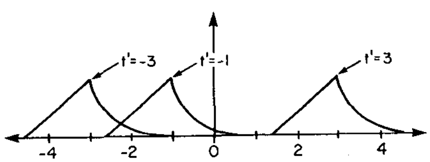
$$z(t)=\int_{-\infty}^{\infty} x(t')h(t-t')\ dt'. \tag{17}$$

This is now called a convolution and is represented by

$$z(t)=x(t)*h(t). \tag{18}$$

The process of convolution can be viewed as flipping one of the two functions, shifting one with respect to the other, multiplying the two and integrating the product for every shift as illustrated by Fig. 2.4.

Fig. 2.3: *The impulse response of a shift invariant filter is shown convolved with three impulses.*



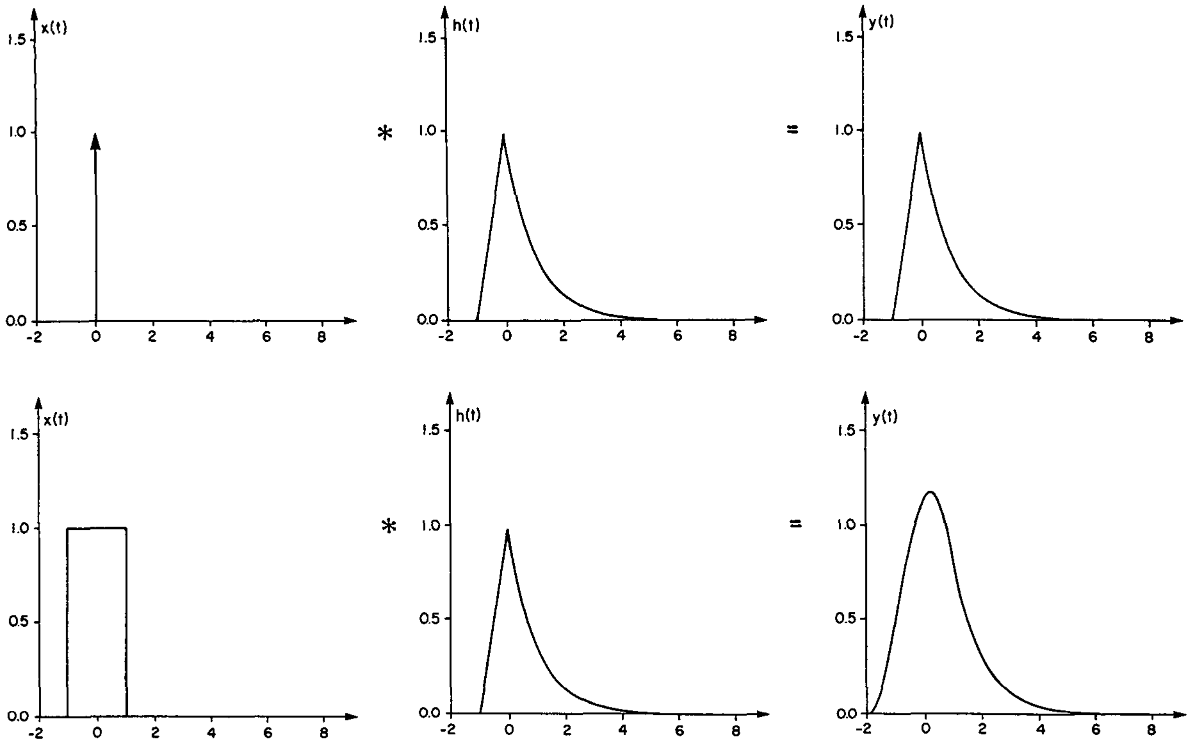


Fig. 2.4: *The results of convolving an impulse response with an impulse (top) and a square pulse (bottom) are shown here.*

Convolution can also be defined for discrete sequences. If

$$x_i = x(i\tau) \tag{19}$$

and

$$y_i = y(i\tau) \tag{20}$$

then the convolution of x_i with y_i can be written as

$$y_i = \tau \sum_{j=-\infty}^{\infty} x_j h_{i-j}. \tag{21}$$

This is a discrete approximation to the integral of (17).

2.1.3 Fourier Representation

For many purposes it is useful to represent functions in the frequency domain. Certainly the most common reason is because it gives a new perspective to an otherwise difficult problem. This is certainly true with the convolution integral; in the time domain convolution is an integral while in the frequency domain it is expressed as a simple multiplication.

In the sections to follow we will describe four different varieties of the Fourier transform. The continuous Fourier transform is mostly used in theoretical analysis. Given that with real world signals it is necessary to periodically sample the data, we are led to three other Fourier transforms that approximate either the time or frequency data as samples of the continuous functions. The four types of Fourier transforms are summarized in Table 2.1.

Assume that we have a continuous function $x(t)$ defined for $T_1 \le t \le T_2$. This function can be expressed in the following form:

$$x(t)=\sum_{k=-\infty}^{\infty} z_k e^{jk\omega_0 t} \tag{22}$$

where $j=\sqrt{-1}$ and $\omega_0 = 2\pi f_0 = 2\pi/T$, $T = T_2 - T_1$ and z_k are complex coefficients to be discussed shortly. What is being said here is that $x(t)$ is the sum of a number of functions of the form

$$e^{jk\omega_0 t}. \tag{23}$$

This function represents

$$e^{jk\omega_0 t}=\cos\ k\omega_0 t+j\ \sin\ k\omega_0 t. \tag{24}$$

The two functions on the right-hand side, commonly referred to as sinusoids, are oscillatory with kf_0 cycles per unit of t as illustrated by Fig. 2.5. kf_0 is

Table 2.1: Four different Fourier transforms can be defined by sampling the time and frequency domains.*

	Continuous Time	Discrete Time
Continuous Frequency	Name: Fourier Transform Forward: $X(\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$ Inverse: $x(t)=1/2\pi \int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$ Periodicity: None	Name: Discrete Fourier Transform Forward: $X(\omega)=\Sigma_{n=-\infty}^{\infty} x(n\tau)e^{-j\omega n\tau}$ Inverse: $x(n\tau)=\tau/2\pi \int_{-\pi/\tau}^{\pi/\tau} X(\omega)e^{j\omega n\tau}\,d\omega$ Periodicity: $X(\omega)=X(\omega+i(2\pi/\tau))$
Discrete Frequency	Name: Fourier Series Forward: $X_n=1/T \int_0^T x(t)e^{-jn(2\pi/T)t}$ Inverse: $x(t)=\Sigma_{n=-\infty}^{\infty} X_n e^{jn(2\pi/T)t}$ Periodicity: $x(t)=x(t+iT)$	Name: Finite Fourier Transform Forward: $X_k=1/N\,\Sigma_{n=0}^{N} x_n e^{-j(2\pi/N)kn}$ Inverse: $x_k=\Sigma_{n=0}^{N} X_n e^{j(2\pi/N)kn}$ Periodicity: $x_k=x_{k+iN}$ and $X_k=X_{k+iN}$

* In the above table time domain functions are indicated by x and frequency domain functions are X. The time domain sampling interval is indicated by τ.

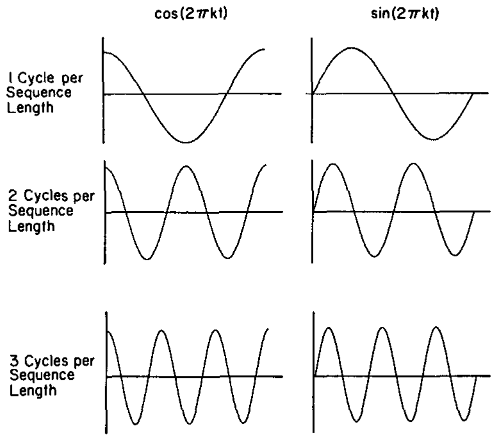


Fig. 2.5: *The first three components of a Fourier series are shown. The cosine waves represent the real part of the signal while the sine waves represent the imaginary.*

called the frequency of the sinusoids. Note that the sinusoids in (24) are at multiples of the frequency f_0, which is called the fundamental frequency.

The coefficients z_k in (22) are called the complex amplitude of the kth component, and can be obtained by using the following formula

$$z_k = \frac{1}{T} \int_{T_1}^{T_2} x(t) e^{-jk\omega_0 T}. \tag{25}$$

The representation in (22) is called the Fourier Series. To illustrate pictorially the representation in (22), we have shown in Fig. 2.6, a triangular function and some of the components from the expansion.

A continuous signal $x(t)$ defined for t between $-\infty$ and ∞ also possesses another Fourier representation called the continuous Fourier transform and defined by

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt. \tag{26}$$

One can show that this relationship may be inverted to yield

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega. \tag{27}$$

Comparing (22) and (27), we see that in both representations, $x(t)$ has been expressed as a sum of sinusoids, $e^{j\omega t}$; the difference being that in the former, the frequencies of the sinusoids are at multiples of ω_0, whereas in the latter we have all frequencies between $-\infty$ to ∞. The two representations are not independent of each other. In fact, the series representation is contained in the continuous transform representation since z_k's in (25) are similar to $X(\omega)$ in (26) for $\omega = k\omega_0 = k(2\pi/T)$, especially if we assume that $x(t)$ is zero outside $[T_1, T_2]$, in which case the range of integration in (27) can be cut

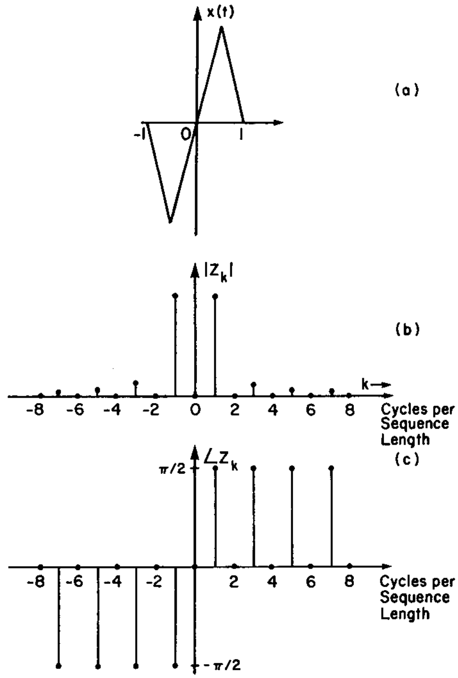


Fig. 2.6: *This illustrates the Fourier series for a simple waveform. A triangle wave is shown in (a) with the magnitude (b) and phase (c) of the first few terms of the Fourier series.*

down to $[T_1, T_2]$. For the case when $x(t)$ is zero outside $[T_1, T_2]$, the reader might ask that since one can recover $x(t)$ from z_k using (22), why use (27) since we require $X(\omega)$ at frequencies in addition to $k\omega_0$'s. The information in $X(\omega)$ for $\omega \neq k\omega_0$ is necessary to constrain the values of $x(t)$ *outside* the interval $[T_1, T_2]$.

If we compute z_k's using (25), and then reconstruct $x(t)$ from z_k's using (22), we will of course obtain the correct values of $x(t)$ within $[T_1, T_2]$; however, if we insist on carrying out this reconstruction outside $[T_1, T_2]$, we will obtain *periodic replications* of the original $x(t)$ (see Fig. 2.7). On the other hand, if $X(\omega)$ is used for reconstructing the signal, we will obtain $x(t)$ within $[T_1, T_2]$ and zero everywhere outside.

The continuous Fourier transform defined in (26) may not exist unless $x(t)$ satisfies certain conditions, of which the following are typical [Goo68]:

1) $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$.
2) $g(t)$ must have only a finite number of discontinuities and a finite number of maxima and minima in any finite interval.
3) $g(t)$ must have no infinite discontinuities.

Some useful mathematical functions, like the Dirac δ function, do not obey the preceding conditions. But if it is possible to represent these functions as limits of a sequence of well-behaved functions that do obey these conditions then the Fourier transforms of the members of this sequence will also form a

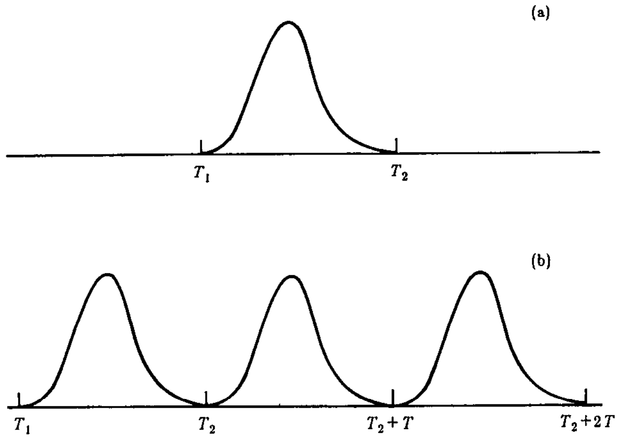


Fig. 2.7: *The signal represented by a Fourier series is actually a periodic version of the original signal defined between T_1 and T_2. Here the original function is shown in (a) and the replications caused by the Fourier series representation are shown in (b).*

sequence. Now if this sequence of Fourier transforms possesses a limit, then this limit is called the "generalized Fourier transform" of the original function. Generalized transforms can be manipulated in the same manner as the conventional transforms, and the distinction between the two is generally ignored; it being understood that when a function fails to satisfy the existence conditions and yet is said to have a transform, then the generalized transform is actually meant [Goo68], [Lig60].

Various transforms described in this section obey many useful properties; these will be shown for the two-dimensional case in Section 2.2.4. Given a relationship for a function of two variables, it is rather easy to suppress one and visualize the one-dimensional case; the opposite is usually not the case.

2.1.4 Discrete Fourier Transform (DFT)

As in the continuous case, a discrete function may also be given a frequency domain representation:

$$X(\omega)=\sum_{n=-\infty}^{\infty} x(n\tau)e^{-j\omega n\tau} \tag{28}$$

where $x(n\tau)$ are the samples of some continuous function $x(t)$, and $X(\omega)$ the frequency domain representation for the sampled data. (*In this book we will generally use lowercase letters to represent functions of time or space and the uppercase letters for functions in the frequency domain.*)

Note that our strategy for introducing the frequency domain representation is opposite of that in the preceding subsection. In describing Fourier series we defined the inverse transform (22), and then described how to compute its coefficients. Now for the DFT we have first described the transform from time into the frequency domain. Later in this section we will describe the inverse transform.

As will be evident shortly, $X(\omega)$ represents the complex amplitude of the sinusoidal component $e^{j\omega\tau n}$ of the discrete signal. Therefore, with one important difference, $X(\omega)$ plays the same role here as z_k in the preceding subsection; the difference being that in the preceding subsection the frequency domain representation was discrete (since it only existed at multiples of the fundamental frequency), while the representation here is continuous as $X(\omega)$ is defined for all ω.

For example, assume that

$$x(n\tau) = \begin{cases} 1 & n=0 \\ -1 & n=1 \\ 0 & \text{elsewhere.} \end{cases} \tag{29}$$

For this signal

$$X(\omega) = 1 - e^{-j\omega\tau}. \tag{30}$$

Note that $X(\omega)$ obeys the following periodicity

$$X(\omega) = X\left(\omega + \frac{2\pi}{\tau}\right) \tag{31}$$

which follows from (28) by simple substitution. In Fig. 2.8 we have shown several periods of this $X(\omega)$.

$X(\omega)$ is called the discrete Fourier transform of the function $x(n\tau)$. From the DFT, the function $x(n\tau)$ can be recovered by using

$$x(n\tau) = \frac{\tau}{2\pi} \int_{-\pi/\tau}^{\pi/\tau} X(\omega) e^{j\omega n\tau} \, d\omega \tag{32}$$

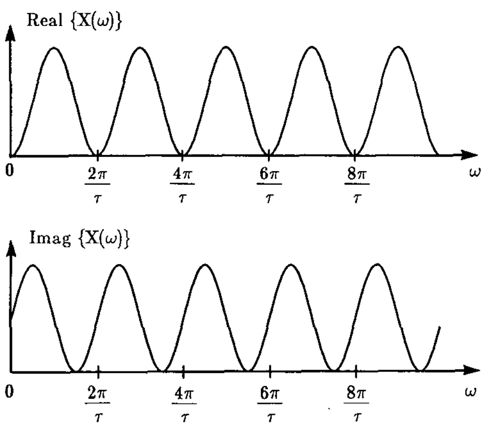


Fig. 2.8: *The discrete Fourier transform (DFT) of a two element sequence is shown here.*

which points to the discrete function $x(n\tau)$ being a sum (an integral sum, to be more specific) of sinusoidal components like $e^{j\omega n\tau}$.

An important property of the DFT is that it provides an alternate method for calculating the convolution in (21). Given a pair of sequences $x_i = x(i\tau)$ and $h_i = h(i\tau)$, their convolution as defined by

$$y_i = \sum_{j=-\infty}^{\infty} x_j h_{i-j}, \tag{33}$$

can be calculated from

$$Y(\omega) = X(\omega)H(\omega). \tag{34}$$

This can be derived by noting that the DFT of the convolution is written as

$$Y(\omega) = \sum_{i=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_k h_{i-k} \right] e^{-j\omega i\tau}. \tag{35}$$

Rewriting the exponential we find

$$Y(\omega) = \sum_{i=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_k h_{i-k} \right] e^{-j\omega(i-k+k)\tau}. \tag{36}$$

The second summation now can be written as

$$Y(\omega) = \sum_{i=-\infty}^{\infty} x_k e^{-j\omega k\tau} \sum_{m=-\infty}^{\infty} h_m e^{-j\omega m\tau}. \tag{37}$$

Note that the limits of the summation remain from $-\infty$ to ∞. At this point it is easy to see that

$$Y(\omega) = X(\omega)H(\omega). \tag{38}$$

A dual to the above relationship can be stated as follows. Let's multiply two discrete functions, x_n and y_n, each obtained by sampling the corresponding continuous function with a sampling interval of τ and call the resulting sequence z_n

$$z_n = x_n y_n. \tag{39}$$

Then the DFT of the new sequence is given by the following convolution in the frequency domain

$$Z(\omega) = \frac{\tau}{2\pi} \int_{-\pi/\tau}^{\pi/\tau} X(\alpha) Y(\omega - \alpha) \, d\alpha. \tag{40}$$

2.1.5 Finite Fourier Transform

Consider a discrete function

$$x(0),\ x(\tau),\ x(2\tau),\ \cdots,\ x((N-1)\tau) \tag{41}$$

that is N elements long. Let's represent this sequence with the following subscripted notation

$$x_0,\ x_1,\ x_2,\ \cdots\ x_{N-1}. \tag{42}$$

Although the DFT defined in Section 2.1.4 is useful for many theoretical discussions, for practical purposes it is the following transformation, called the finite Fourier transform (FFT),[1] that is actually calculated with a computer:

$$X_u = \frac{1}{N} \sum_{n=0}^{N-1} x_n e^{-j(2\pi/N)un} \tag{43}$$

for $u = 0, 1, 2, \cdots, N - 1$. To explain the meaning of the values X_u, rewrite (43) as

$$X\left(u \frac{1}{N\tau}\right) = \frac{1}{N} \sum_{n=0}^{N-1} x(n\tau) e^{-j2\pi(u(1/N\tau))(n\tau)}. \tag{44}$$

Comparing (44) and (28), we see that the X_u's are the samples of the continuous function $X(\omega)$ for

$$\omega = u \frac{1}{N\tau} \qquad \text{with } u = 0,\ 1,\ 2,\ \cdots,\ N-1. \tag{45}$$

Therefore, we see that if (43) is used to compute the frequency domain representation of a discrete function, a sampling interval of τ in the t-domain implies a sampling interval of $1/N\tau$ in the frequency domain. The inverse of the relationship shown in (43) is

$$x_n = \sum_{u=0}^{N-1} X_u e^{j(2\pi/N)un}, \qquad n = 0,\ 1,\ 2,\ \cdots,\ N-1. \tag{46}$$

Both (43) and (46) define sequences that are periodically replicated. First consider (43). If the $u = Nm + i$ term is calculated then by noting that $e^{j(2\pi/N)Nm} = 1$ for all integer values of m, it is easy to see that

$$X_{Nm+i} = X_i. \tag{47}$$

[1] The acronym FFT also stands for fast Fourier transform, which is an efficient algorithm for the implementation of the finite Fourier transform.

A similar analysis can be made for the inverse case so that

$$x_{Nm+i} = x_i. \tag{48}$$

When the finite Fourier transforms of two sequences are multiplied the result is still a convolution, as it was for the discrete Fourier transform defined in Section 2.1.4, but now the convolution is with respect to replicated sequences. This is often known as circular convolution because of the effect discussed below.

To see this effect consider the product of two finite Fourier transforms. First write the product of two finite Fourier transforms

$$Z_u = X_u Y_u \tag{49}$$

and then take the inverse finite Fourier transform to find

$$z_n = \sum_{u=0}^{N-1} e^{j(2\pi/N)un} X_u Y_u. \tag{50}$$

Substituting the definition of X_u and Y_u as given by (43) the product can now be written

$$z_n = \frac{1}{N^2} \sum_{u=0}^{N-1} e^{j(2\pi/N)un} \sum_{i=0}^{N-1} x_i e^{j(2\pi/N)iu} \sum_{k=0}^{N-1} y_k e^{j(2\pi/N)ku}. \tag{51}$$

The order of summation can be rearranged and the exponential terms combined to find

$$z_n = \frac{1}{N^2} \sum_{i=0}^{N-1} \sum_{k=0}^{N-1} x_i y_k \sum_{u=0}^{N-1} e^{j(2\pi/N)un-ui-uk}. \tag{52}$$

There are two cases to consider. When $n - i - k \neq 0$ then as a function of u the samples of the exponential $e^{j(2\pi/N)un-ui-uk}$ represent an integral number of cycles of a complex sinusoid and their sum is equal to zero. On the other hand, when $i = n - k$ then each sample of the exponential is equal to one and thus the summation is equal to N. The summation in (52) over i and k represents a sum of all the possible combinations of x_i and y_k. When $i = n - k$ then the combination is multiplied by a factor of N while when $i \neq n - k$ then the term is ignored. This means that the original product of two finite Fourier transforms can be simplified to

$$z_n = \frac{1}{N} \sum_{k=0}^{N-1} x_{n-k} y_k. \tag{53}$$

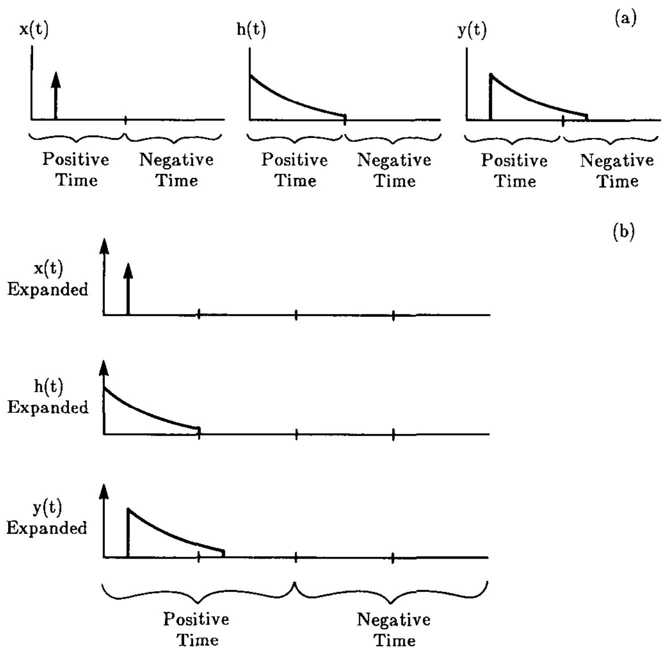
This expression is very similar to (21) except for the definition of x_{n-k} and y_k for negative indices. Consider the case when $n = 0$. The first term of the summation is equal to $x_0 y_0$ but the second term is equal to $x_{-1} y_1$. Although in the original formulation of the finite Fourier transform, the x sequence was only specified for indices from 0 through $N - 1$, the periodicity property in (48) implies that x_{-1} be equal to x_{N-1}. This leads to the name circular convolution since the undefined portions of the original sequence are replaced by a circular replication of the original data.

The effect of circular convolution is shown in Fig. 2.9(a). Here we have shown an exponential sequence convolved with an impulse. The result represents a circular convolution and not samples of the continuous convolution.

A circular convolution can be turned into an aperiodic convolution by zero-padding the data. As shown in Fig. 2.9(b) if the original sequences are doubled in length by adding zeros then the original N samples of the product sequence will represent an aperiodic convolution of the two sequences.

Efficient procedures for computing the finite Fourier transform are known as fast Fourier transform (FFT) algorithms. To calculate each of the N points of the summation shown in (43) requires on the order of N^2 operations. In a fast Fourier transform algorithm the summation is rearranged to take advantage of common subexpressions and the computational expense is reduced to $N \log N$. For a 1024 point signal this represents an improvement by a factor of approximately 100. The fast Fourier transform algorithm has revolutionized digital signal processing and is described in more detail in [Bri74].

Fig. 2.9: *The effect of circular convolution is shown in (a). (b) shows how the data can be zero-padded so that when an FFT convolution is performed the result represents samples of an aperiodic convolution.*



2.1.6 Just How Much Data Is Needed?

In Section 2.1.1 we used a sequence of numbers x_i to approximate a continuous function $x(t)$. An important question is, how finely must the data be sampled for x_i to accurately represent the original signal? This question was answered by Nyquist who observed that a signal must be sampled at least twice during each cycle of the highest frequency of the signal. More rigorously, if a signal $x(t)$ has a Fourier transform such that

$$X(\omega)=0 \qquad \text{for } \omega \geq \frac{\omega_N}{2} \tag{54}$$

then samples of x must be measured at a rate greater than ω_N. In other words, if T is the interval between consecutive samples, we want $2\pi/T \geq \omega_N$. The frequency ω_N is known as the Nyquist rate and represents the minimum frequency at which the data can be sampled without introducing errors.

Since most real world signals aren't limited to a small range of frequencies, it is important to know the consequences of sampling at below the Nyquist rate. We can consider the process of sampling to be equivalent to multiplication of the original continuous signal $x(t)$ by a sampling function given by

$$h(t)=\sum_{-\infty}^{\infty} \delta(t-iT). \tag{55}$$

The Fourier transform of $h(t)$ can be computed from (26) to be

$$H(\omega)=\left(\frac{2\pi}{T}\right)\sum_{-\infty}^{\infty} \delta\left(\omega-\frac{2\pi i}{T}\right). \tag{56}$$

By (40) we can convert the multiplication to a convolution in the frequency domain. Thus the result of the sampling can be written

$$Z(\omega)=\left(\frac{2\pi}{T}\right)\sum_{i=-\infty}^{\infty} X\left(\omega-\frac{2i\pi}{T}\right). \tag{57}$$

This result is diagrammed in Fig. 2.10.

It is important to realize that when sampling the original data (Fig. 2.10(a)) at a rate faster than that defined by the Nyquist rate, the sampled data are an exact replica of the original signal. This is shown in Fig. 2.10(b). If the sampled signal is filtered such that all frequencies above the Nyquist rate are removed, then the original signal will be recovered.

On the other hand, as the sampling interval is increased the replicas of the signal in Fig. 2.10(c) move closer together. With a sampling interval greater than that predicted by the Nyquist rate some of the information in the original data has been smeared by replications of the signal at other frequencies and the original signal is unrecoverable. (See Fig. 2.10(d).) The error caused by the sampling process is given by the inverse Fourier transform of the frequency information in the overlap as shown in Fig. 2.10(d). These errors are also known as aliasing.

2.1.7 Interpretation of the FFT Output

Correct interpretation of the X_u's in (43) is obviously important. Toward that goal, it is immediately apparent that X_0 stands for the average (or, what is more frequently called the dc) component of the discrete function, since from (43)

$$X_0 = \frac{1}{N} \sum_{n=0}^{N-1} x_n. \tag{58}$$

Interpretation of X_1 requires, perhaps, a bit more effort; it stands for 1 cycle per sequence length. This can be made obvious by setting $X_1 = 1$, while all

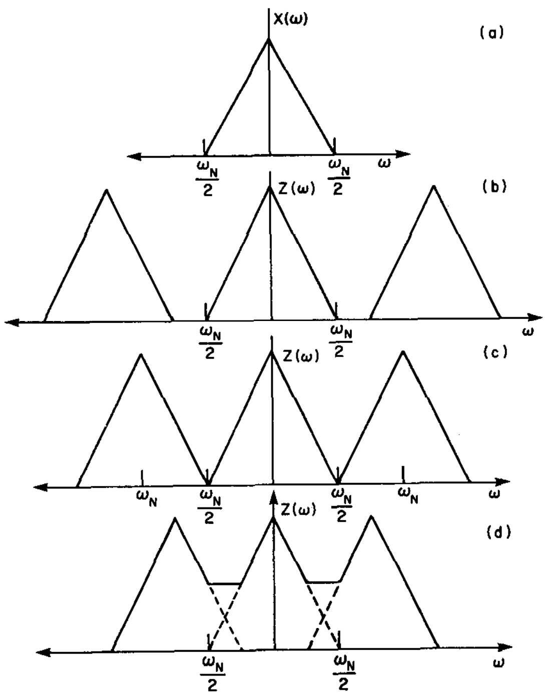


Fig. 2.10: *Sampling a waveform generates replications of the original Fourier transform of the object at periodic intervals. If the signal is sampled at a frequency of ω then the Fourier transform of the object will be replicated at intervals of 2ω. (a) shows the Fourier transform of the original signal, (b) shows the Fourier transform when x(t) is sampled at a rate faster than the Nyquist rate, (c) when sampled at the Nyquist rate and finally (d) when the data are sampled at a rate less than the Nyquist rate.*

other X_i's are set equal to 0 in (46). We obtain

$$x_n = e^{j2(\pi/N)n} \tag{59}$$

$$= \cos\left(\frac{2\pi}{N}n\right) + j \sin\left(\frac{2\pi}{N}n\right) \tag{60}$$

for $n = 0, 1, 2, \cdots, N-1$. A plot of either the cosine or the sine part of this expression will show just one cycle of the discrete function x_n, which is why we consider X_1 as representing one cycle per sequence length. One may similarly show that X_2 represents two cycles per sequence length. Unfortunately, this straightforward approach for interpreting X_u breaks down for $u > N/2$. For these high values of the index u, we make use of the following periodicity property

$$X_{-u} = X_{N-u} \tag{61}$$

which is easily proved by substitution in (43). For further explanation, consider now a particular value for N, say 8. We already know that

X_0 represents dc
X_1 represents 1 cycle per sequence length
X_2 represents 2 cycles per sequence length
X_3 represents 3 cycles per sequence length
X_4 represents 4 cycles per sequence length.

From the periodicity property we can now add the following

X_5 represents -3 cycles per sequence length
X_6 represents -2 cycles per sequence length
X_7 represents -1 cycle per sequence length.

Note that we could also have added "X_4 represents -4 cycles per sequence length." The fact is that for any N element sequence, $X_{N/2}$ will always be equal to $X_{-N/2}$, since from (43)

$$X_{N/2} = X_{-N/2} = \sum_{0}^{N-1} x_n(-1)^n. \tag{62}$$

The discussion is diagrammatically represented by Fig. 2.11, which shows that when an N element data sequence is fed into an FFT program, the output sequence, also N elements long, consists of the dc frequency term, followed by positive frequencies and then by negative frequencies. This type of an output where the negative axis information follows the positive axis information is somewhat unnatural to look at.

To display the FFT output with a more natural progression of frequencies, we can, of course, rearrange the output sequence, although if the aim is

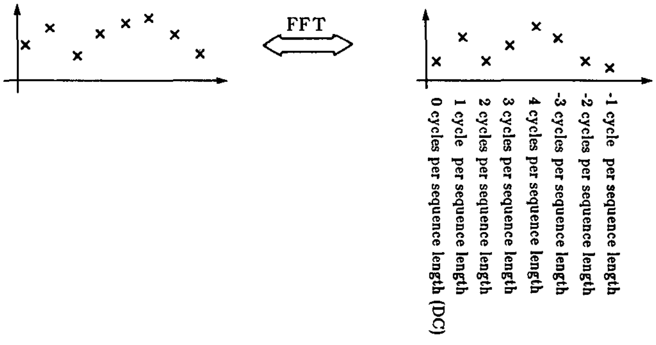


Fig. 2.11: *The output of an 8 element FFT is shown here.*

merely to filter the data, it may not be necessary to do so. In that case the filter transfer function can be rearranged to correspond to the frequency assignments of the elements of the FFT output.

It is also possible to produce normal-looking FFT outputs (with dc at the center between negative and positive frequencies) by "modulating" the data prior to taking the FFT. Suppose we multiply the data with $(-1)^n$ to produce a new sequence x'_n

$$x'_n = x_n(-1)^n. \tag{63}$$

Let X'_u designate the FFT of this new sequence. Substituting (63) in (43), we obtain

$$X'_u = X_{u-N/2} \tag{64}$$

for $u = 0, 1, 2, \cdots, N-1$. This implies the following equivalences

$$X'_0 = X_{-N/2} \tag{65}$$

$$X'_1 = X_{-N/2+1} \tag{66}$$

$$X'_2 = X_{-N/2+2} \tag{67}$$

$$\vdots \tag{68}$$

$$X'_{N/2} = X_0 \tag{69}$$

$$\vdots \tag{70}$$

$$X'_{N-1} = X_{N/2-1}. \tag{71}$$

2.1.8 How to Increase the Display Resolution in the Frequency Domain

The right column of Fig. 2.12 shows the magnitude of the FFT output (the dc is centered) of the sequence that represents a rectangular function as shown in the left column. As was mentioned before, the Fourier transform of a discrete sequence contains all frequencies, although it is periodic, and the FFT output represents the samples of one period. For many situations, the

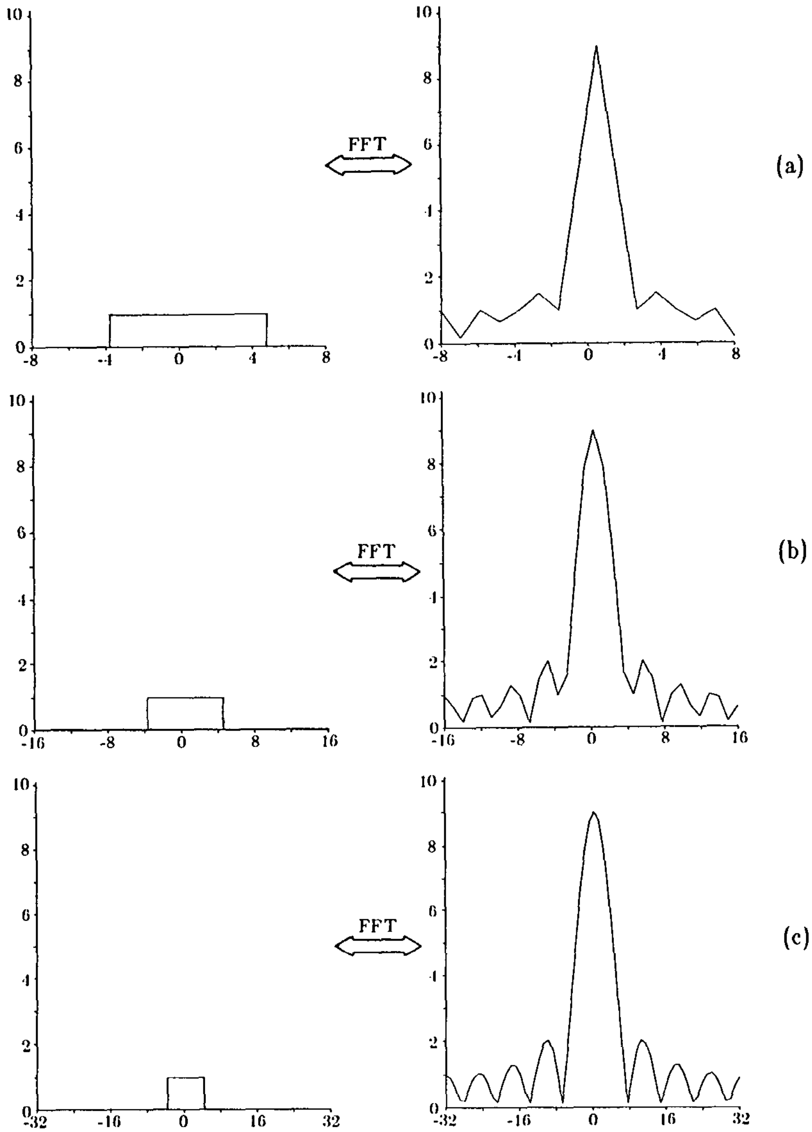


Fig. 2.12: *As shown here, padding a sequence of data with zeros increases the resolution in the frequency domain. The sequence in (a) has only 16 points, (b) has 32 points, while (c) has 64 points.*

frequency domain samples supplied by the FFT, although containing practically all the information for the reconstruction of the continuous Fourier transform, are hard to interpret visually. This is evidenced by Fig. 2.12(a), where for part of the display we have only one sample associated with an oscillation in the frequency domain. It is possible to produce smoother-looking outputs by what is called zero-padding the data before taking the FFT. For example, if the sequence of Fig. 2.12(a) is extended with zeros to twice its length, the FFT of the resulting 32 element sequence will be as shown in Fig. 2.12(b), which is visually smoother looking than the pattern in Fig. 2.12(a). If we zero-pad the data to four times its original length, the output is as shown in Fig. 2.12(c).

That zero-padding a data sequence yields frequency domain points that are more closely spaced can be shown by the following derivation. Again let x_1, x_2, $\cdots$, x_{N-1} represent the original data. By zero-padding the data we will define a new x' sequence:

$$x_n' = x_n \qquad \text{for } n=0, 1, 2, \cdots, N-1 \tag{72}$$

$$= 0 \qquad \text{for } n=N, N+1, \cdots, 2N-1. \tag{73}$$

Let X_u' be the FFT of the new sequence x_n'. Therefore,

$$X_u' = \sum_0^{2N-1} x_n' e^{-j(2\pi/2N)un} \tag{74}$$

which in terms of the original data is equal to

$$X_u' = \sum_0^{N-1} x_n e^{-j(2\pi/2N)un}. \tag{75}$$

If we evaluate this expression at even values of u, that is when

$$u = 2m \qquad \text{where } m=0, 1, 2, \cdots, N-1 \tag{76}$$

we get

$$X_{2m}' = \sum_0^{N-1} x_n e^{-j(2\pi/N)mn} \tag{77}$$

$$= X_m. \tag{78}$$

In Fig. 2.13 is illustrated the equality between the even-numbered elements of the new transform and the original transform. That X_1', X_3', $\cdots$, etc. are the interpolated values between X_0 and X_1; between X_1 and X_2; etc. can be seen from the summations in (43) and (74) written in the following form

$$X_u' = X'\left(m \frac{2\pi}{2N\tau}\right) = \sum_{n=0}^{N-1} x(n\tau) e^{-j((2\pi/2N\tau)m)n\tau} \tag{79}$$

$$X_m = X\left(m \frac{2\pi}{N\tau}\right) = \sum_{n=0}^{N-1} x(n\tau) e^{-j(2\pi m/N\tau)n\tau}. \tag{80}$$

Comparing the two summations, we see that the upper one simply represents the sampled DFT with half the sampling interval.

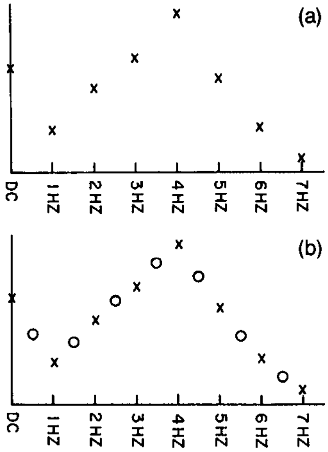


Fig. 2.13: *When a data sequence is padded with zeros the effect is to increase the resolution in the frequency domain. The points in (a) are also in the longer sequence shown in (b), but there are additional points, as indicated by circles, that provide interpolated values of the FFT.*

So we have the following conclusion: to increase the display resolution in the frequency domain, we must zero-extend the time domain signal. This also means that if we are comparing the transforms of sequences of *different* lengths, they must all be zero-extended to the *same* number, so that they are all plotted with the same display resolution. This is because the upper summation, (79), has a sampling interval in the frequency domain of $2\pi/2N\tau$ while the lower summation, (80), has a sampling interval that is twice as long or $2\pi/N\tau$.

2.1.9 How to Deal with Data Defined for Negative Time

Since the forward and the inverse FFT relationships, (43) and (46), are symmetrical, the periodicity property described in (62) also applies in time domain. What is being said here is that if a time domain sequence and its transform obey (43) and (46), then an N element data sequence in the time domain must satisfy the following property

$$x_{-n} = x_{N-n}. \tag{81}$$

To explain the implications of this property, consider the case of $N = 8$, for which the data sequence may be written down as

$$x_0,\ x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ x_7. \tag{82}$$

By the property under discussion, this sequence should be interpreted as

$$x_0,\ x_1,\ x_2,\ x_3,\ x_4 \text{ (or } x_{-4}),\ x_{-3},\ x_{-2},\ x_{-1}. \tag{83}$$

Then if our data are defined for negative indices (times), and, say, are of the following form

$$x_{-3},\ x_{-2},\ x_{-1},\ x_0,\ x_1,\ x_2,\ x_3,\ x_4 \tag{84}$$

they should be fed into an FFT program as

$$x_0,\ x_1,\ x_2,\ x_3,\ x_4,\ x_{-3},\ x_{-2},\ x_{-1}. \tag{85}$$

To further drive home the implications of the periodicity property in (62), consider the following example, which consists of taking an 8 element FFT of the data

$$0.9 \quad 0.89 \quad 0.88 \quad 0.87 \quad 0.86 \quad 0.85 \quad 0.84 \quad 0.83. \tag{86}$$

We insist for the sake of explaining a point, that only an 8 element FFT be taken. If the given data have no association with time, then the data should be fed into the program as they are presented. However, if it is definitely known that the data are ZERO before the first element, then the sequence presented to the FFT program should look like

$$\underbrace{0.9 \quad 0.89 \quad 0.88 \quad 0.87}_{\text{positive time}} \quad \underbrace{\frac{0.86+0}{2} \quad 0 \quad 0 \quad 0.}_{\text{negative time}} \tag{87}$$

positive time (88)

negative time (89)

This sequence represents the given fact that at $t = -1, -2$ and -3 the data are supposed to be zero. Also, since the fifth element represents both x_4 and x_{-4} (these two elements are supposed to be equal for ideal data), and since in the given data the element x_{-4} is zero, we simply replace the fifth element by the average of the two. Note that in the data fed into the FFT program, the sharp discontinuity at the origin, as represented by the transition from 0 to 0.9, has been retained. This discontinuity will contribute primarily to the high frequency content of the transform of the signal.

2.1.10 How to Increase Frequency Domain Display Resolution of Signals Defined for Negative Time

Let's say that we have an eight element sequence of data defined for both positive and negative times as follows:

$$x_{-3}\ x_{-2}\ x_{-1}\ x_0\ x_1\ x_2\ x_3\ x_4. \tag{90}$$

It can be fed into an FFT algorithm after it is rearranged to look like

$$x_0\ x_1\ x_2\ x_3\ x_4\ x_{-3}\ x_{-2}\ x_{-1}. \tag{91}$$

If x_{-4} was also defined in the original sequence, we have three options: we can either ignore x_{-4}, or ignore x_4 and retain x_{-4} for the fifth from left position in the above sequence, or, better yet, use $(x_{-4} + x_4)/2$ for the fifth position. Note we are making use of the property that due to the data periodicity properties assumed by the FFT algorithm, the fifth element corresponds to both x_4 and x_{-4} and in the ideal case they are supposed to be equal to each other.

Now suppose we wish to double the display resolution in the frequency domain; we must then zero-extend the data as follows

$$x_0\ x_1\ x_2\ x_3\ x_4\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ x_{-4}\ x_{-3}\ x_{-2}\ x_{-1}. \tag{92}$$

Note that we have now given separate identities to x_4 and x_{-4}, since they don't have to be equal to each other anymore. So if they are separately available, they can be used as such.

2.1.11 Data Truncation Effects

To see the data truncation effects, consider a signal defined for all indices n. If $X(\omega)$ is the true DFT of this signal, we have

$$X(\omega)=\sum_{-\infty}^{\infty} x_n e^{-j\omega n T_s}. \tag{93}$$

Suppose we decide to take only a 16 element transform, meaning that of all the x_n's, we will retain only 16.

Assuming that the most significant transitions of the signal occur in the base interval defined by n going from -7 to 8, we may write approximately

$$X(\omega)\simeq\sum_{-7}^{8} x_n e^{-j\omega n T_s}. \tag{94}$$

More precisely, if $X'(\omega)$ denotes the DFT of the truncated data, we may write

$$X'(\omega)=\sum_{-7}^{8} x_n e^{-j\omega n T_s} \tag{95}$$

$$=\sum_{-\infty}^{\infty} x_n I_{16}(n) e^{-j\omega n T_s} \tag{96}$$

where $I_{16}(n)$ is a function that is equal to 1 for n between -7 and 8, and zero outside. By the convolution theorem

$$X'(\omega)=\frac{T_s}{2\pi}\, X(\omega) * A(\omega) \tag{97}$$

where

$$A(\omega)=\sum_{-7}^{8} e^{-j\omega n T_s} \tag{98}$$

$$=e^{-j\omega(T_s/2)} \frac{\sin \frac{\omega N T_s}{2}}{\sin \frac{\omega T_s}{2}} \tag{99}$$

with $N = 16$. This function is displayed in Fig. 2.14, and illustrates the nature of distortion introduced by data truncation.

2.2 Image Processing

The signal processing concepts described in the first half of this chapter are easily extended to two dimensions. As was done before, we will describe how to represent an image with delta functions, linear operations on images and the use of the Fourier transform.

2.2.1 Point Sources and Delta Functions

Let O be an operation that takes pictures into pictures; given the input picture f, the result of applying O to f is denoted by $O[f]$. Like the 1-dimensional case discussed earlier in this chapter, we call O *linear* if

$$O[af+bg]=aO[f]+bO[g] \tag{100}$$

for all pictures, f, g and all constants a, b.

In the analysis of linear operations on pictures, the concept of a *point*

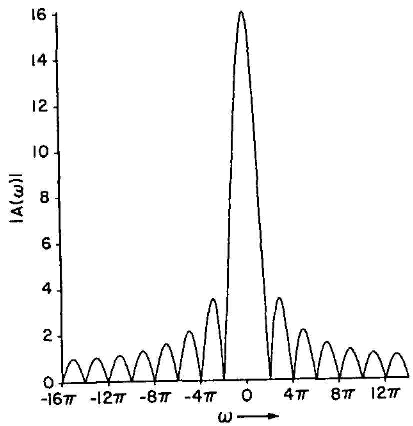


Fig. 2.14: *Truncating a sequence of data is equivalent to multiplying it by a rectangular window. The result in the frequency domain is to convolve the Fourier transform of the signal with the window shown above.*

source is very convenient. If any arbitrary picture f could be considered to be a sum of point sources, then a knowledge of the operation's output for a point source input could be used to determine the output for f. Whereas for one-dimensional signal processing the response due to a point source input is called the impulse response, in image processing it is usually referred to as the *point spread function* of O. If in addition the point spread function is not dependent on the location of the point source input then the operation is said to be space invariant.

A point source can be regarded as the limit of a sequence of pictures whose nonzero values become more and more concentrated spatially. Note that in order for the total brightness to be the same for each of these pictures, their nonzero values must get larger and larger. As an example of such a sequence of pictures, let

$$\text{rect}\ (x,\ y)=\begin{cases}1 & \text{for } |x|\le\frac{1}{2} \text{ and } |y|\le\frac{1}{2}\\ 0 & \text{elsewhere}\end{cases} \tag{101}$$

(see Fig. 2.15) and let

$$\delta_n(x,\ y)=n^2\ \text{rect}\ (nx,\ ny), \qquad n=1,\ 2,\ \cdots . \tag{102}$$

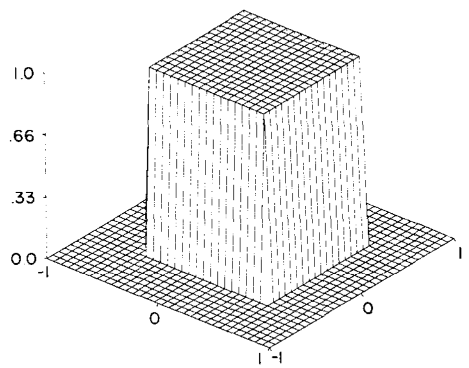
Thus δ_n is zero outside the $1/n \times 1/n$ square described by $|x| \le 1/2n$, $|y| \le 1/2n$ and has constant value n^2 inside that square. It follows that

$$\iint_{-\infty}^{\infty} \delta_n(x,\ y)\ dx\ dy=1 \tag{103}$$

for any n.

As $n \to \infty$, the sequence δ_n does not have a limit in the usual sense, but it is convenient to treat it as though its limit existed. This limit, denoted by δ, is

Fig. 2.15: *As in the one-dimensional case, the delta function (δ) is defined as the limit of the rectangle function shown here.*



called a *Dirac delta function*. Evidently, we have $\delta(x, y) = 0$ for all (x, y) other than (0, 0) where it is infinite. It follows that $\delta(-x, -y) = \delta(x, y)$.

A number of the properties of the one-dimensional delta function described earlier extend easily to the two-dimensional case. For example, in light of (103), we can write

$$\iint_{-\infty}^{\infty} \delta(x, y)\, dx\, dy = 1. \tag{104}$$

More generally, consider the integral $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)\delta_n(x, y)\, dx\, dy$. This is just the average of $g(x, y)$ over a $1/n \times 1/n$ square centered at the origin. Thus in the limit we retain just the value at the origin itself, so that we can conclude that the area under the delta function is one and write

$$\iint_{-\infty}^{\infty} g(x, y)\delta(x, y)\, dx\, dy = g(0, 0). \tag{105}$$

If we shift δ by the amount (α, β), i.e., we use $\delta(x - \alpha, y - \beta)$ instead of $\delta(x, y)$, we similarly obtain the value of g at the point (α, β), i.e.,

$$\iint_{-\infty}^{\infty} g(x, y)\delta(x-\alpha, y-\beta)\, dx\, dy = g(\alpha, \beta). \tag{106}$$

The same is true for any region of integration containing (α, β). Equation (106) is called the "sifting" property of the δ function.

As a final useful property of δ, we have

$$\iint_{-\infty}^{\infty} \exp\,[-j2\pi(ux+vy)]\, du\, dv = \delta(x, y). \tag{107}$$

For a discussion of this property, see Papoulis [Pap62].

2.2.2 Linear Shift Invariant Operations

Again let us consider a linear operation on images. The point spread function, which is the output image for an input point source at the origin of the xy-plane, is denoted by $h(x, y)$.

A linear operation is said to be *shift invariant* (or space invariant, or position invariant) if the response to $\delta(x - \alpha, y - \beta)$, which is a point source located at (α, β) in the xy-plane, is given by $h(x - \alpha, y - \beta)$. In other words, the output is merely shifted by α and β in the x and y directions, respectively.

Now let us consider an arbitrary input picture $f(x, y)$. By (106) this picture can be considered to be a linear sum of point sources. We can write $f(x, y)$ as

$$f(x, y)=\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\alpha, \beta)\delta(\alpha-x, \beta-y)\, d\alpha\, d\beta. \tag{108}$$

In other words, the image $f(x, y)$ is a linear sum of point sources located at (α, β) in the xy-plane with α and β ranging from $-\infty$ to $+\infty$. In this sum the point source at a particular value of (α, β) has "strength" $f(\alpha, \beta)$. Let the response of the operation to the input $f(x, y)$ be denoted by $O[f]$. If we assume the operation to be shift invariant, then by the interpretation just given to the right-hand side of (108), we obtain

$$O[f(x, y)]=O\left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\alpha, \beta)\delta(\alpha-x, \beta-y)\, d\alpha\, d\beta\right] \tag{109}$$

$$=\iint f(\alpha, \beta)O[\delta(\alpha-x, \beta-y)]\, d\alpha\, d\beta \tag{110}$$

by the linearity of the operation, which means that the response to a sum of excitations is equal to the sum of responses to each excitation. As stated earlier, the response to $\delta(\alpha - x, \beta - y)$ $[=\delta(x - \alpha, y - \beta)]$, which is a point source located at (α, β), is given by $h(x - \alpha, y - \beta)$ and if $O[f]$ is denoted by g, we obtain

$$g(x, y)=\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\alpha, \beta)h(x-\alpha, y-\beta)\, d\alpha\, d\beta. \tag{111}$$

The right-hand side is called the *convolution* of f and h, and is often denoted by $f * h$. The integrand is a product of two functions $f(\alpha, \beta)$ and $h(\alpha, \beta)$ with the latter rotated about the origin by 180° and shifted by x and y along the x and y directions, respectively. A simple change of variables shows that (111) can also be written as

$$g(x, y)=\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-\alpha, y-\beta)h(\alpha, \beta)\, d\alpha\, d\beta \tag{112}$$

so that $f * h = h * f$.

Fig. 2.16 shows the effect of a simple blurring operation on two different images. In this case the point response, h, is given by

$$h(x, y)=\begin{cases} 1 & x^2+y^2<0.25^2 \\ 0 & \text{elsewhere.} \end{cases} \tag{113}$$

As can be seen in Fig. 2.16 one effect of this convolution is to smooth out the edges of each image.

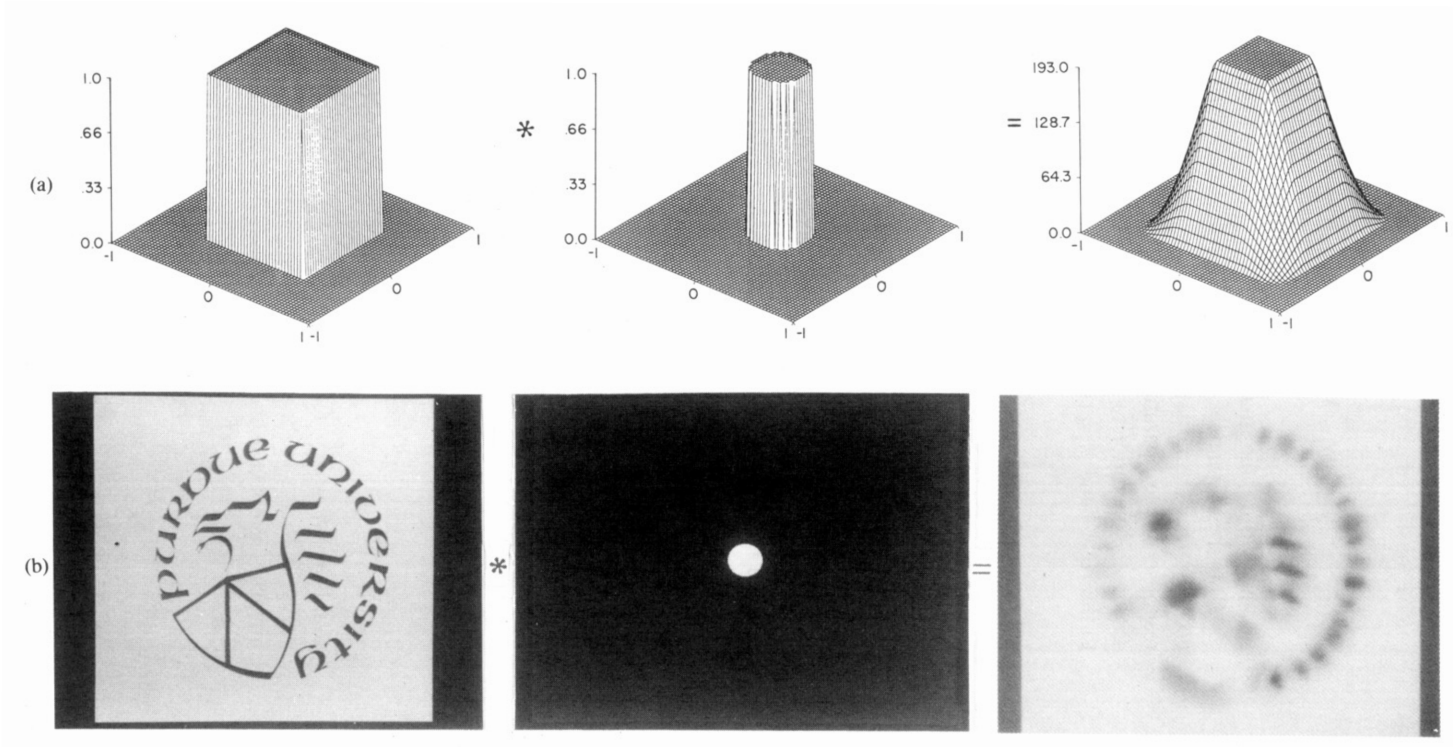


Fig. 2.16: *The two-dimensional convolutions of a circular point spread function and a square (a) and a binary image (b) are shown.*

2.2.3 Fourier Analysis

Representing two-dimensional images in the Fourier domain is as useful as it is in the one-dimensional case. Let $f(x, y)$ be a function of two independent variables x and y; then its *Fourier transform* $F(u, v)$ is defined by

$$F(u, v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)e^{-j2\pi(ux+vy)}\, dx\, dy. \tag{114}$$

In the definition of the one- and two-dimensional Fourier transforms we have used slightly different notations. Equation (26) represents the frequency in terms of radians per unit length while the above equation represents frequency in terms of cycles per unit length. The two forms are identical except for a scaling and either form can be converted to the other using the relation

$$f=u=v=2\pi\omega. \tag{115}$$

By splitting the exponential into two halves it is easy to see that the two-dimensional Fourier transform can be considered as two one-dimensional transforms; first with respect to x and then y

$$F(u, v)=\int_{-\infty}^{\infty} e^{-j2\pi vy}\, dy \int_{-\infty}^{\infty} f(x, y)e^{-j2\pi ux}\, dx. \tag{116}$$

In general, F is a complex-valued function of u and v. As an example, let $f(x, y) = \text{rect}\ (x, y)$. Carrying out the integration indicated in (114) we find

$$F(u, v)=\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} e^{-j2\pi(ux+vy)}\, dx\, dy \tag{117}$$

$$=\frac{\sin\ \pi u}{\pi u}\int_{-1/2}^{1/2} e^{-j2\pi vy}\, dy \tag{118}$$

$$=\frac{\sin\ \pi u}{\pi u}\frac{\sin\ \pi v}{\pi v}. \tag{119}$$

This last function is usually denoted by sinc (u, v) and is illustrated in Fig. 2.17. More generally, using the change of variables $x' = nx$ and $y' = ny$, it is easy to show that the Fourier transform of rect (nx, ny) is

$$(1/n^2)\ \text{sinc}\ (u/n,\ v/n). \tag{120}$$

Given the definition of the Dirac delta function as a limit of a sequence of the functions n^2 rect (nx, ny); by the arguments in Section 2.1.3, the Fourier transform of the Dirac delta function is the limit of the sequence of Fourier

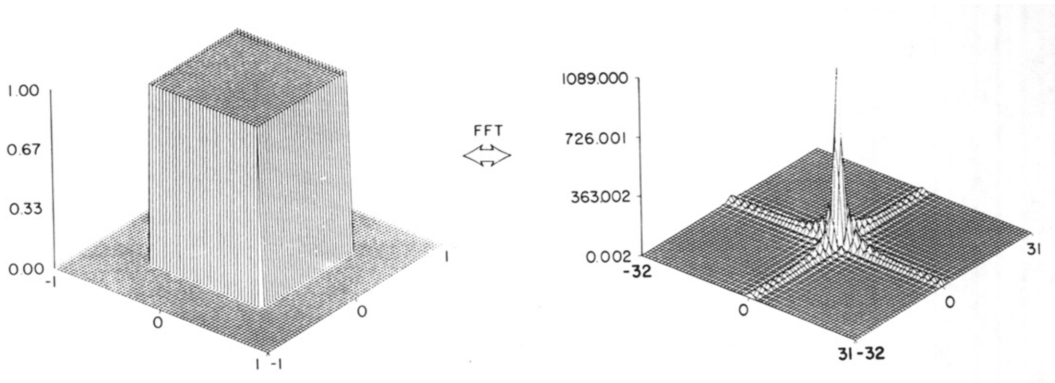


Fig. 2.17: *The two-dimensional Fourier transform of the rectangle function is shown here.*

transforms sinc $(u/n, v/n)$. In other words, when

$$f(x, y)=\delta(x, y) \tag{121}$$

then

$$F(u, v)=\lim_{n\to\infty} \text{sinc } (u/n, v/n)=1. \tag{122}$$

The inverse Fourier transform of $F(u, v)$ is found by multiplying both sides of (114) by $e^{j2\pi(ux+v\beta)}$ and integrating with respect to u and v to find

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u, v) \exp [-j2\pi(ux+v\beta)] \, du \, dv$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)e^{j2\pi(u\alpha+v\beta)}e^{j2\pi(ux+v\beta)} \, du \, dv \, dx \, dy \tag{123}$$

or

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)e^{-j2\pi[u(x-\alpha)+v(y-\beta)]} \, du \, dv \, dx \, dy. \tag{124}$$

Making use of (107) it is easily shown that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u, v) \exp [j2\pi(ux+v\beta)] \, du \, dv$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\delta(x-\alpha, y-\beta) \, dx \, dy \tag{125}$$

$$=f(\alpha, \beta) \tag{126}$$

or equivalently

$$f(x, y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u, v) \exp [j2\pi(ux+vy)] \, du \, dv. \tag{127}$$

This integral is called the *inverse Fourier transform* of $F(u, v)$. By (114) and (127), $f(x, y)$ and $F(u, v)$ form a *Fourier transform pair.*

If x and y represent spatial coordinates, (127) can be used to give a physical interpretation to the Fourier transform $F(u, v)$ and to the coordinates u and v. Let us first examine the function

$$e^{j2\pi(ux+vy)}. \tag{128}$$

The real and imaginary parts of this function are $\cos 2\pi(ux + vy)$ and $\sin 2\pi(ux + vy)$, respectively. In Fig. 2.18(a), we have shown $\cos 2\pi(ux + vy)$. It is clear that if one took a section of this two-dimensional pattern parallel to the x-axis, it goes through u cycles per unit distance, while a section parallel to the y-axis goes through v cycles per unit distance. This is the reason why u and v are called the *spatial frequencies* along the x- and y-axes, respectively. Also, from the figure it can be seen that the spatial period of the pattern is $(u^2 + v^2)^{-1/2}$. The plot for $\sin 2\pi(ux + vy)$ looks similar to the one in Fig. 2.18(a) except that it is displaced by a quarter period in the direction of maximum rate of change.

From the preceding discussion it is clear that $e^{j2\pi(ux+vy)}$ is a two-dimensional pattern, the sections of which, parallel to the x- and y-axes, are spatially periodic with frequencies u and v, respectively. The pattern itself has a spatial period of $(u^2 + v^2)^{-1/2}$ along a direction that subtends an angle $\tan^{-1}(v/u)$ with the x-axis. By changing u and v, one can generate patterns with spatial periods ranging from 0 to ∞ in any direction in the xy-plane.

Equation (127) can, therefore, be interpreted to mean that $f(x, y)$ is a linear combination of elementary periodic patterns of the form $e^{j2\pi(ux+vy)}$. Evidently, the function, $F(u, v)$, is simply a weighting factor that is a measure of the relative contribution of the elementary pattern to the total sum. Since u and v are the spatial frequency of the pattern in the x and y directions, $F(u, v)$ is called the *frequency spectrum* of $f(x, y)$.

2.2.4 Properties of Fourier Transforms

Several properties of the two-dimensional Fourier transform follow easily from the defining integrals equation. Let $F\{f\}$ denote the Fourier transform of a function $f(x, y)$. Then $F\{f(x, y)\} = F(u, v)$. We will now present without proof some of the more common properties of Fourier transforms. The proofs are, for the most part, left for the reader (see the books by Goodman [Goo68] and Papoulis [Pap62]).

1) Linearity:

$$F\{af_1(x, y)+bf_2(x, y)\}=aF\{f_1(x, y)\}+bF\{f_2(x, y)\} \tag{129}$$

$$=aF_1(u, v)+bF_2(u, v). \tag{130}$$

This follows from the linearity of the integration operation.

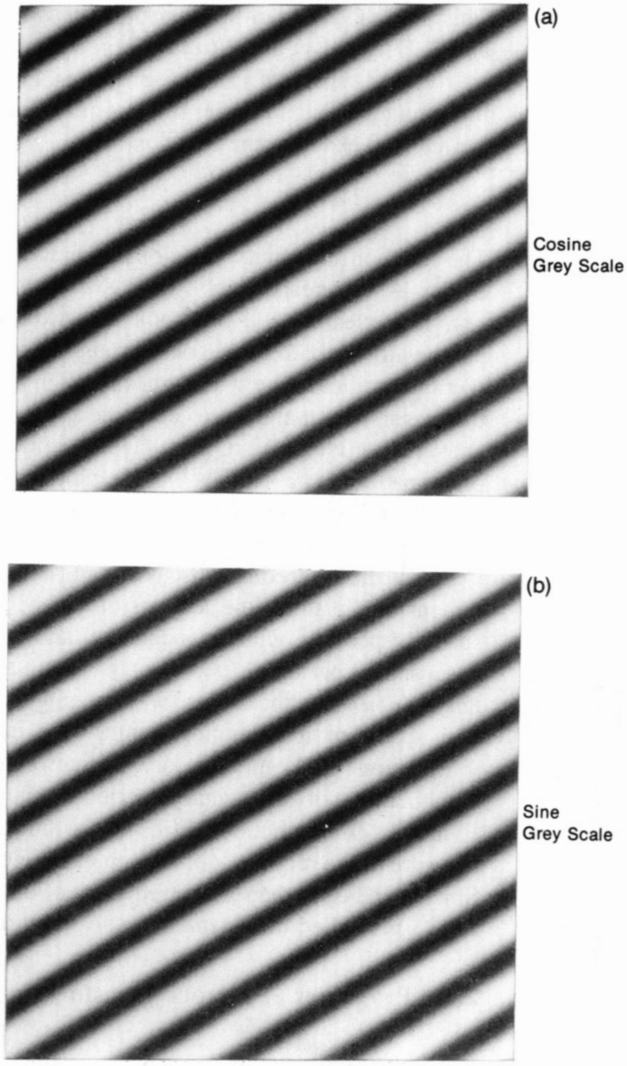


Fig. 2.18: *The Fourier transform represents an image in terms of exponentials of the form* $e^{j2\pi(ux+vy)}$. *Here we have shown the real (cosine) and the imaginary (sine) parts of one such exponential.*

2) Scaling:

$$F\{f(\alpha x,\ \beta y)\} = \frac{1}{|\alpha\beta|} F\left(\frac{u}{\alpha},\ \frac{v}{\beta}\right). \tag{131}$$

To see this, introduce the change of variables $x' = \alpha x$, $y' = \beta y$. This property is illustrated in Fig. 2.19.

3) Shift Property:

$$F\{f(x-\alpha,\ y-\beta)\} = F(u,\ v)e^{-j2\pi(u\alpha+v\beta)}. \tag{132}$$

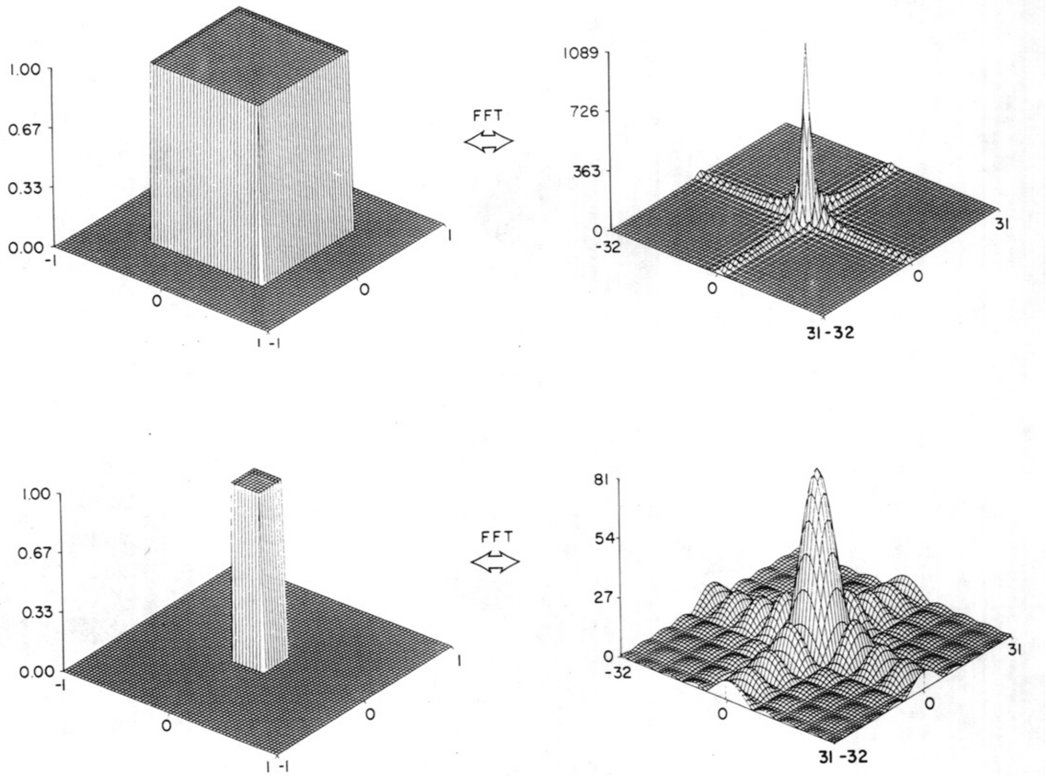


Fig. 2.19: *Scaling the size of an image leads to compression and amplification in the Fourier domain.*

This too follows immediately if we make the change of variables $x' = x - \alpha$, $y' = y - \beta$. The corresponding property for a shift in the frequency domain is

$$F\ \{\exp\ [j2\pi(u_0x+v_0y)]f(x,\ y)\}=F(u-u_0,\ v-v_0). \tag{133}$$

4) Rotation by Any Angle: In polar coordinates we can write

$$F\{f(r,\ \theta)\}=F(\omega,\ \phi). \tag{134}$$

If the function, f, is rotated by an angle α then the following result follows

$$F\{f(r,\ \theta+\alpha)\}=F(\omega,\ \phi+\alpha). \tag{135}$$

This property is illustrated in Fig. 2.20.

5) Rotational Symmetry: If $f(x, y)$ is a circularly symmetric function, i.e., $f(r, \theta)$ is only a function of r, then its frequency spectrum is also

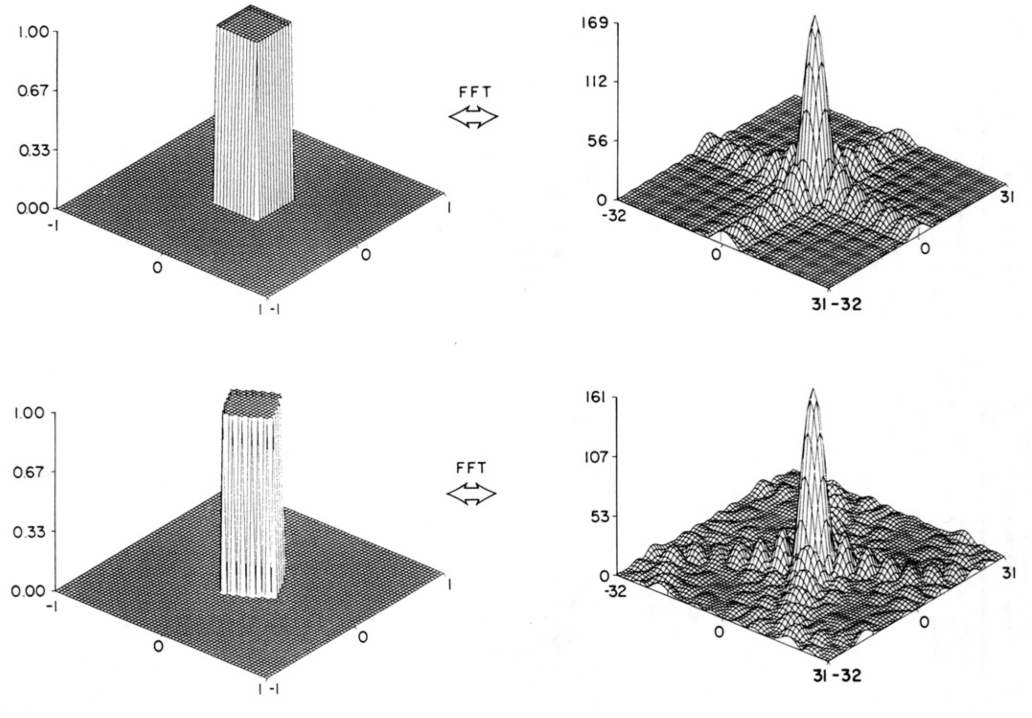


Fig. 2.20: *Rotation of an object by 30° leads to a similar rotation in the Fourier transform of the image.*

circularly symmetric and is given by

$$F(u, v)=F(p)=2\pi \int_0^\infty rf(r)J_0(2\pi rp)\ dr. \tag{136}$$

The inverse relationship is given by

$$f(r)=2\pi \int_0^\infty pF(p)J_0(2\pi rp)\ dp \tag{137}$$

where

$$r=\sqrt{x^2+y^2},\ 0=\tan^{-1}(y/x),\ p=\sqrt{u^2+v^2},\ \phi=\tan^{-1}(v/u) \tag{138}$$

and

$$J_0(x)=(1/2\pi)\int_0^{2\pi} \exp\ [-jx \cos(\theta-\phi)]\ d\theta \tag{139}$$

is the zero-order Bessel function of the first kind. The transformation in (136) is also called the *Hankel transform of zero order*.

6) 180° Rotation:

$$F\{F\{f(x, y)\}\} = f(-x, -y). \tag{140}$$

7) Convolution:

$$F\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(\alpha, \beta)f_2(x-\alpha, y-\beta)\, d\alpha\, d\beta\right\}$$

$$= F\{f_1(x, y)\}F\{f_2(x, y)\} \tag{141}$$

$$= F_1(u, v)F_2(u, v). \tag{142}$$

Note that the convolution of two functions in the space domain is equivalent to the very simple operation of multiplication in the spatial frequency domain. The corresponding property for convolution in the spatial frequency domain is given by

$$F\{f_1(x, y)f_2(x, y)\} = \iint_{-\infty}^{\infty} F_1(u-s, v-t)F_2(s, t)\, ds\, dt. \tag{143}$$

A useful example of this property is shown in Figs. 2.21 and 2.22. By the Fourier convolution theorem we have chosen a frequency domain function, H, such that all frequencies above Ω cycles per picture are zero. In the space domain the convolution of x and h is a simple linear filter while in the frequency domain it is easy to see that all frequency components above Ω cycles/picture have been eliminated.

8) Parseval's Theorem:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(x, y)f_2^*(x, y)\, dx\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(u, v)F_2^*(u, v)\, du\, dy \tag{144}$$

where the asterisk denotes the complex conjugate. When $f_1(x, y) = f_2(x, y) = f(x, y)$, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x, y)|^2\, dx\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(u, v)|^2\, du\, dv. \tag{145}$$

In this form, this property is interpretable as a statement of conservation of energy.

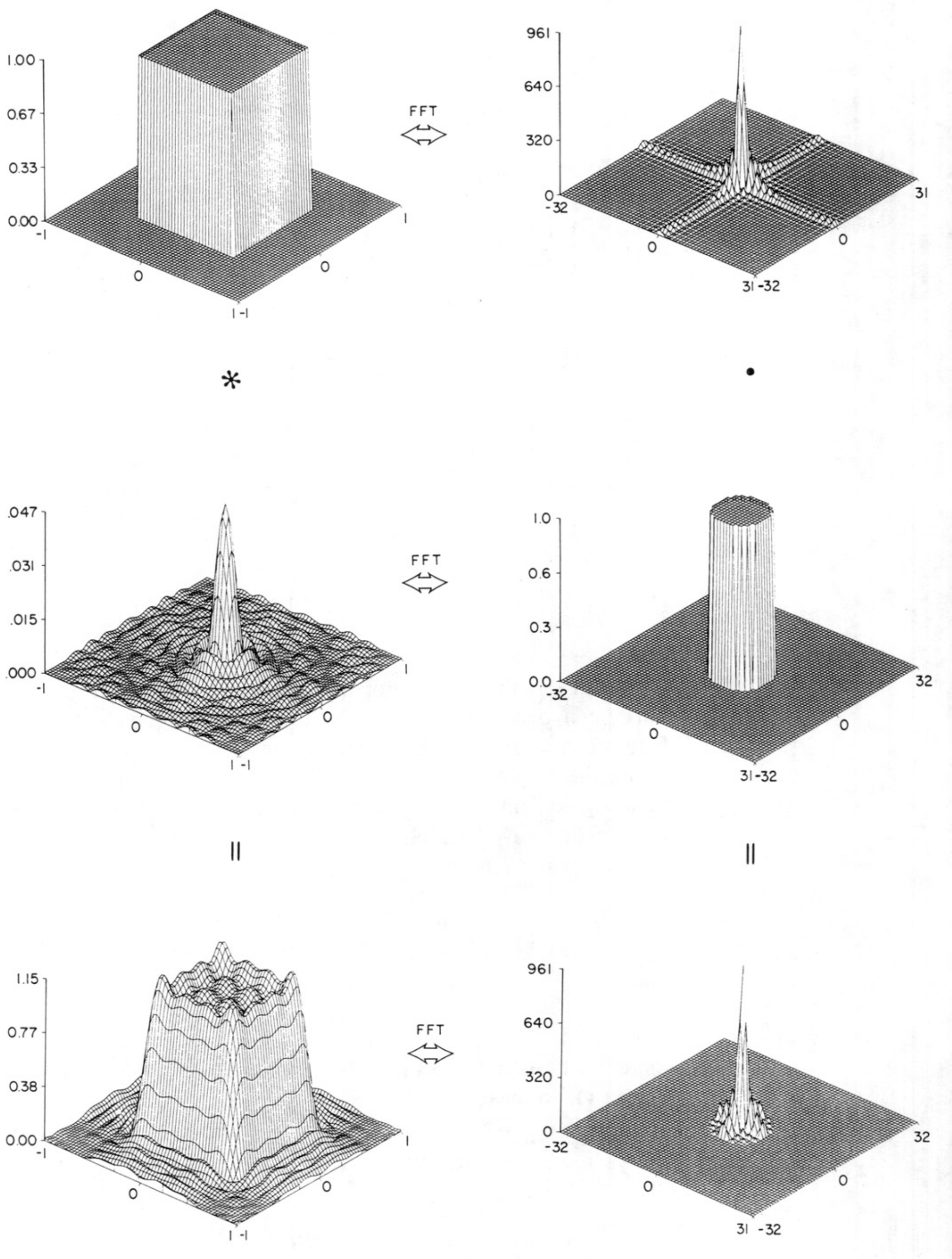


Fig. 2.21: *An ideal low pass filter is implemented by multiplying the Fourier transform of an object by a circular window.*

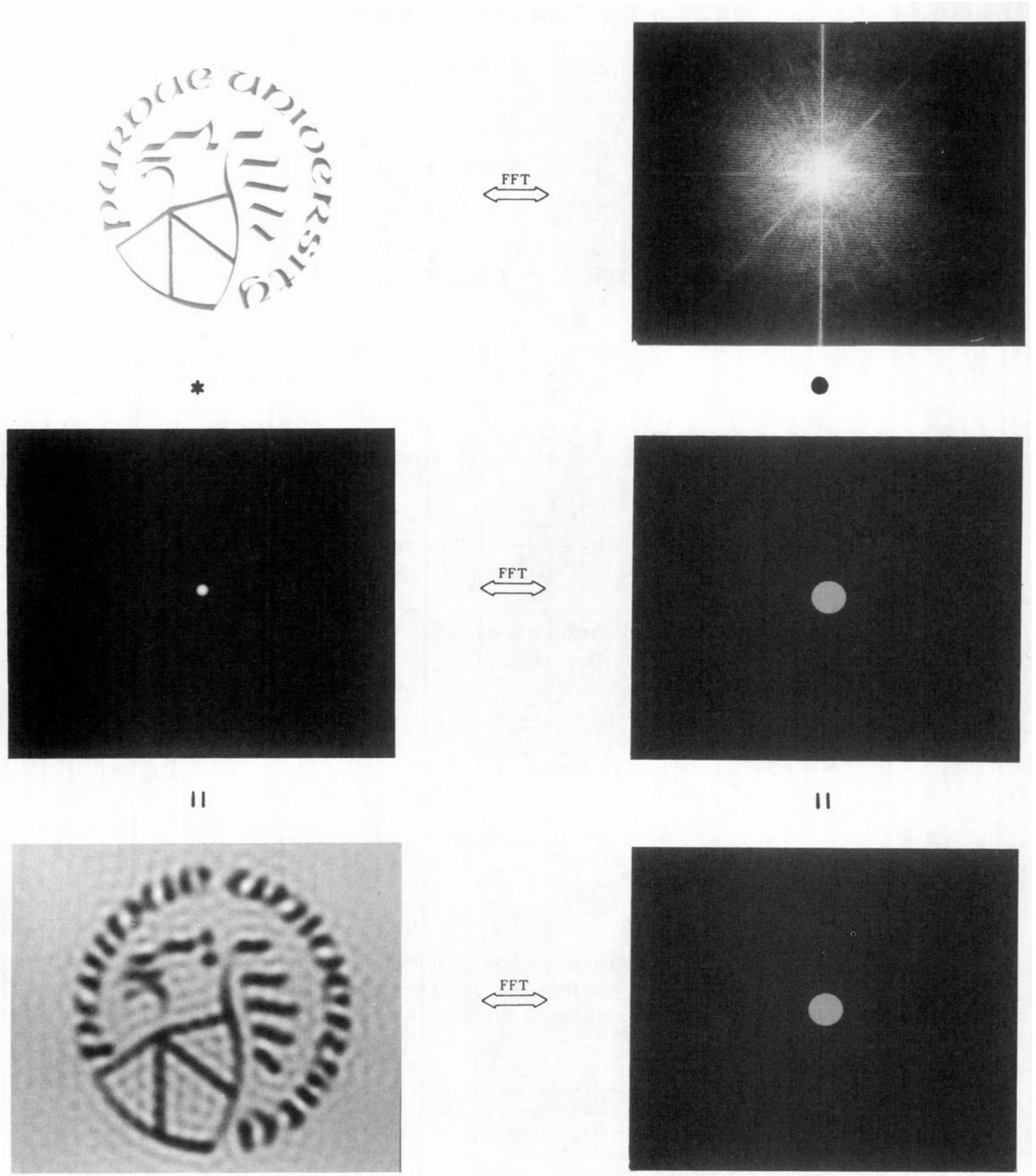


Fig. 2.22: *An ideal low pass filter is implemented by multiplying the Fourier transform of an object by a circular window.*

2.2.5 The Two-Dimensional Finite Fourier Transform

Let $f(m, n)$ be a sampled version of a continuous two-dimensional function f. The finite Fourier transform (FFT) is defined as the summation[2]

$$F(u, v)=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m, n) \exp\left[-j2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)\right] \tag{146}$$

for $u = 0, 1, 2, \cdots, M-1$; $v = 0, 1, 2, \cdots, N-1$.

The inverse FFT (IFFT) is given by the summation

$$f(m, n)=\sum_{u=0}^{M-1}\sum_{r=0}^{N-1} F(u, v) \exp\left[j2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)\right] \tag{147}$$

for $m = 0, 1, \cdots, M-1$; $n = 0, 1, \cdots, N-1$. It is easy to verify that the summations represented by the FFT and IFFT are inverses by noting that

$$\sum_{m=0}^{J-1} \exp\left[\frac{-j2\pi}{J} km\right] \exp\left[\frac{j2\pi}{J} mn\right]=\begin{cases} J, & k=n \\ 0, & k\neq n. \end{cases} \tag{148}$$

This is the discrete version of (107). That the inverse FFT undoes the effect of the FFT is seen by substituting (43) into (147) for the inverse DFT to find

$$f(m, n)=\frac{1}{MN}\sum_{u=0}^{M-1}\sum_{r=0}^{N-1}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m, n)$$

$$\cdot \exp\left[-j2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)\right] \exp\left[j2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)\right]. \tag{149}$$

The desired result is made apparent by rearranging the order of summation and using (148).

In (146) the discrete Fourier transform $F(u, v)$ is defined for u between 0 and $M-1$ and for v between 0 and $N-1$. If, however, we use the same equation to evaluate $F(\pm u, \pm v)$, we discover that the periodicity properties

[2] To be consistent with the notation in the one-dimensional case, we should express the space and frequency domain arrays as $f_{m,n}$ and $F_{u,v}$. However, we feel that for the two-dimensional case, the math looks a bit neater with the style chosen here, especially when one starts dealing with negative indices and other extensions. Also, note that the variables u and v are indices here, which is contrary to their usage in Section 2.2.3 where they represent continuously varying spatial frequencies.

of the exponential factor imply that

$$F(u, -v)=F(u, N-v) \tag{150}$$

$$F(-u, v)=F(M-u, v) \tag{151}$$

$$F(-u, -v)=F(M-u, N-v). \tag{152}$$

Similarly, using (147) we can show that

$$f(-m, n)=f(M-m, n) \tag{153}$$

$$f(m, -n)=f(m, N-n) \tag{154}$$

$$f(-m, -n)=f(M-m, N-n). \tag{155}$$

Another related consequence of the periodicity properties of the exponential factors in (28) and (147) is that

$$F(aM+u, bN+v)=F(u, v) \text{ and } f(aM+m, bN+n)=f(m, n) \tag{156}$$

for $a = 0, \pm 1, \pm 2, \cdots, b = 0, \pm 1, \pm 2, \cdots$. Therefore, we have the following conclusion: if a finite array of numbers $f_{m,n}$ and its Fourier transform $F_{u,v}$ are related by (28) and (147), then if it is desired to extend the definition of $f(m, n)$ and $F(u, v)$ beyond the original domain as given by $[0 \leq (m \text{ and } u) \leq M - 1]$ and $[0 \leq (n \text{ and } v) \leq N - 1]$, this extension must be governed by (151), (154) and (156). In other words, the extensions are periodic repetitions of the arrays.

It will now be shown that this periodicity has important consequences when we compute the convolution of two $M \times N$ arrays, $f(m, n)$ and $d(m, n)$, by multiplying their finite Fourier transforms, $F(u, v)$ and $D(u, v)$. The convolution of two arrays $f(m, n)$ and $d(m, n)$ is given by

$$g(\alpha, \beta)=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m, n)d(\alpha-m, \beta-n) \tag{157}$$

$$=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(\alpha-m, \beta-n)d(m, n) \tag{158}$$

for $\alpha = 0, 1, \cdots, M - 1, \beta = 0, 1, \cdots, N - 1$, where we insist that when the values of $f(m, n)$ and $d(m, n)$ are required for indices outside the ranges $0 \leq m \leq M - 1$ and $0 \leq n \leq N - 1$, for which $f(m, n)$ and $d(m, n)$ are defined, then they should be obtained by the rules given in (151), (154) and (156). With this condition, the convolution previously defined becomes a circular or cyclic convolution.

As in the 1-dimensional case, the FFT of (157) can be written as the product of the two Fourier transforms. By making use of (147), we obtain

$$g(\alpha, \beta)=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m, n)d(\alpha-m, \beta-n) \tag{159}$$

expanding f and d in terms of their DFTs

$$=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left\{\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u, v) \exp\left[j2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)\right]\right\}$$

$$\cdot\left\{\sum_{w=0}^{M-1}\sum_{z=0}^{N-1} D(w, z) \exp\left[j2\pi\left((\alpha-m)\frac{w}{M}+\frac{(\beta-n)z}{N}\right)\right]\right\} \tag{160}$$

and then rearranging the summations

$$=\frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}\sum_{w=0}^{M-1}\sum_{z=0}^{N-1}\left\{F(u, v)D(w, z) \exp\left[j2\pi\left(\frac{\alpha w}{M}+\frac{\beta z}{N}\right)\right]\right.$$

$$\left.\cdot\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \exp\left[j2\pi\frac{m(u-w)}{M}\right] \exp\left[j2\pi\frac{n(v-z)}{N}\right]\right\}. \tag{161}$$

Using the orthogonality relationship (148) we find

$$=\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u, v)D(u, v) \exp\left[j2\pi\left(\frac{\alpha u}{M}+\frac{\beta v}{N}\right)\right]. \tag{162}$$

Thus we see that the convolution of the two-dimensional arrays f and d can be expressed as a simple multiplication in the frequency domain.

The discrete version of Parseval's theorem is an often used property of the finite Fourier transform. In the continuous case this theorem is given by (144) while for the discrete case

$$\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m, n)g^*(m, n)=MN\sum_{u=0}^{M-1}\sum_{u=0}^{N-1} F(u, v)G^*(u, v). \tag{163}$$

The following relationship directly follows from (163):

$$\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} |f(m, n)|^2=MN\sum_{u=0}^{M-1}\sum_{n=0}^{N-1} |F(u, v)|^2. \tag{164}$$

As in the one-dimensional and the continuous two-dimensional cases Parseval's theorem states that the energy in the space domain and that in the frequency domain are equal.

As in a one-dimensional case, a two-dimensional image must be sampled at a rate greater than the Nyquist frequency to prevent errors due to aliasing. For a moment, going back to the interpretation of u and v as continuous frequencies (see Section 2.2.3), if the Fourier transform of the image is zero for all frequencies greater than B, meaning that $F(u, v) = 0$ for all u and v such that $|u| \geq B$ and $|v| \geq B$, then there will be no aliasing if samples of the image are taken on a rectangular grid with intervals of less than $\frac{1}{2B}$. A pictorial representation of the effect of aliasing on two-dimensional images is shown in Fig. 2.23. Further discussion on aliasing in two-dimensional sampling can be found in [Ros82].

2.2.6 Numerical Implementation of the Two-Dimensional FFT

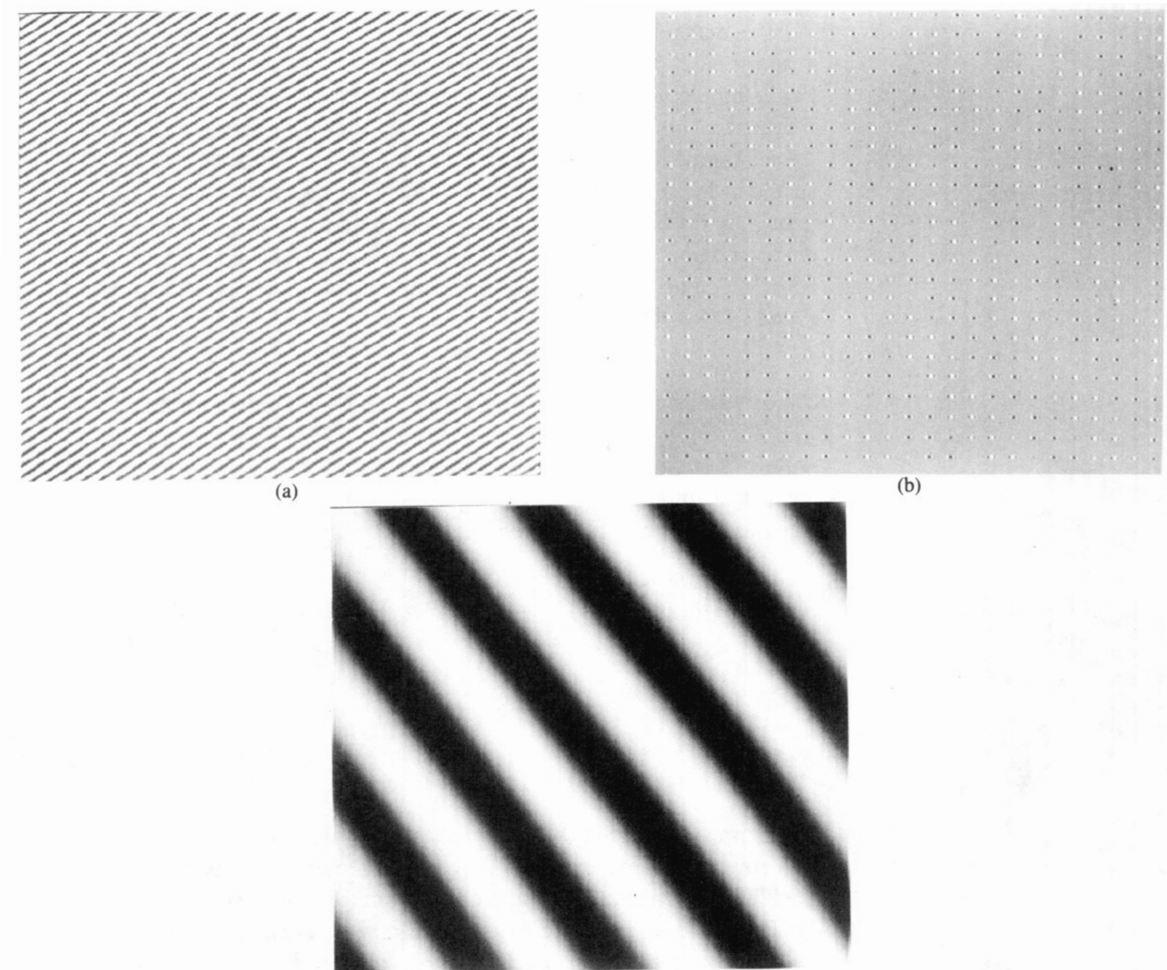
Before we end this chapter, we would like to say a few words about the numerical implementation of the two-dimensional finite Fourier transform. Equation (28) may be written as

$$F(u, v) = \frac{1}{M} \sum_{m=0}^{M-1} \left[\frac{1}{N} \sum_{n=0}^{N-1} f(m, n) \exp\left(-j \frac{2\pi}{N} nv\right) \right] \cdot \exp\left(-j \frac{2\pi}{M} mu\right),$$
$$u=0, \cdots, M-1, \; v=0, \cdots, N-1. \tag{165}$$

The expression within the square brackets is the one-dimensional FFT of the mth row of the image, which may be implemented by using a standard FFT (fast Fourier transform) computer program (in most instances N is a power of 2). Therefore, to compute $F(u, v)$, we *replace each row in the image by its one-dimensional FFT, and then perform the one-dimensional FFT of each column.*

Ordinarily, when a 2-D FFT is computed in the manner described above, the frequency domain origin will not be at the center of the array, which if displayed as such can lead to difficulty in interpretation. Note, for example, that in a 16×16 image the indices $u = 15$ and $v = 0$ correspond to a negative frequency of one cycle per image width. This can be seen by substituting $u = 1$ and $v = 0$ in the second equation in (151). To display the frequency domain origin at approximately the center of the array (a precise center does not exist when either M or N is an even number), the image data $f(m, n)$ are first multiplied by $(-1)^{m+n}$ and then the finite Fourier transformation is performed. To prove this, let us define a new array $f'(m, n)$ as follows:

$$f'(m, n) = f(m, n)(-1)^{m+n} \tag{166}$$



(a) (b) (c)

Fig. 2.23: *The effect of aliasing in two-dimensional images is shown here. (This is often known as the Moiré effect.) In (a) a high-frequency sinusoid is shown. In (b) this sinusoid is sampled at a rate much lower than the Nyquist rate and the sampled values are shown as black and white dots (gray is used to represent the area between the samples). Finally, in (c) the sampled data shown in (b) are low pass filtered at the Nyquist rate. Note that both the direction and frequency of the sinusoid have changed due to aliasing.*

and let $F'(u, v)$ be its finite Fourier transform:

$$F'(u, v)=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m, n)(-1)^{m+n} \cdot \exp\left[-j2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)\right]. \tag{167}$$

Rewriting this expression as

$$F(u, v)=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m, n)$$

$$\cdot \exp\left\{j2\pi\left[\frac{(M/2)m}{M}+\frac{(N/2)n}{N}\right]\right\} \tag{168}$$

$$\cdot \exp\left[-j2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)\right] \tag{169}$$

it is easy to show that

$$F(u,\ v)=F\left(u-\frac{M}{2},\ v-\frac{N}{2}\right),$$
$$u=0,\ 1,\ \cdots,\ M-1;\ v=0,\ 1,\ \cdots,\ N-1. \tag{170}$$

Therefore, when the array $F'(u, v)$ is displayed, the location at $u = M/2$ and $v = N/2$ will contain $F(0, 0)$.

We have by no means discussed all the important properties of continuous, discrete and finite Fourier transforms; the reader is referred to the cited literature for further details.

2.3 References

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