The All-Loop S-Matrix of $\mathcal{N} = 4$ Super Yang-Mills

Jacob L. Bourjaily
Princeton University & IAS

in collaboration with
N. Arkani-Hamed, F. Cachazo, and J. Trnka
also with Andrew Hodges and S. Caron-Huot,

[arXiv:1012.6032], [arXiv:1012.6030], [arXiv:1008.2958],
Outline

1. Spiritus Movens
   - MHV Amplitudes in Quantum Chromodynamics: A Parable
   - The Generalization of Parke-Taylor’s Formula Through 3-Loops

2. Preliminaries: The (Tree-Level) Analytic S-Matrix, Redux
   - Colour & Kinematics: the Vernacular of the S-Matrix
   - Tree-Level Recursion: Making the Impossible, Possible
   - Momentum Twistors and Geometry: Trivializing Kinematics

3. Beyond Trees: Recursion Relations for Loop-Amplitudes
   - The Loop Integrand in Momentum-Twistor Space
   - Pushing BCFW Forward to All-Loop Orders
   - The Geometry of Forward Limits

4. Local Loop Integrals for Scattering Amplitudes
   - Leading Singularities and Schubert Calculus
   - Manifestly-Finite Momentum-Twistor Integrals
   - Pushing the Analytic S-Matrix Forward
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Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist’s, but also a theorist’s delight.
Parke and Taylor’s Heroic Computation: Six Months Later

Six months later, they had come upon a “guess”, not just for not their amplitude but an infinite number of amplitudes!

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\[ A_n^{(2)}(\ldots, j^-, \ldots, k^-, \ldots) = \frac{\langle j \ k \rangle^4}{\langle 1 \ 2 \rangle \langle 2 \ 3 \rangle \cdots \langle n \ 1 \rangle} \]
Generalizing Parke-Taylor’s Formula Through 3-Loops:

In recent months, similar simplifications have been ‘guessed’ (and checked):

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\]

\[
\times \left\{ 1 + \sum_{i<j<i} \left( 1 + \sum_{i<j<k<l<i} \cdots \right) \right\}
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\[ \times \left\{ 1 + \sum_{i<j<i}^{i<i<j<i} + \frac{1}{2} \sum_{i<j<k<l<i}^{i<j<k<l<i} + \frac{1}{3} \sum_{i_1 \leq i_2 < j_1 \leq j_2 \leq k_1 \leq k_2 < i_1} \right\} \]
Simple Sources of Simplification

An $n$-point scattering amplitude is specified by listing each particle’s:

- momentum, (which we take to be incoming)
- helicity
- colour
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By shuffling all colour-factors to the outside of every Feynman diagram, we can write the amplitude* for any desired colour-ordering in terms of any other.

**Colour-ordered partial amplitudes**

$$A_n(\{p_a\}) = \sum \text{Tr}(T^{a_1} \cdots T^{a_n}) A_n(p_{a_1}, \ldots, p_{a_n})$$

*e.g. $A_9(1^+, 2^+, 3^-, 4^+, 5^-, 6^+, 7^-, 8^+, 9^-)$*
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Scattering amplitudes for massless particles are not directly functions four-momenta, but functions of spinor variables:

$$p_a^\mu \rightarrow p_a^{\alpha \dot{\alpha}} \equiv p_a^\mu \sigma_\mu^{\alpha \dot{\alpha}} = \left( \begin{array}{ccc} p_0^a + p_3^a & p_1^a - ip_2^a \\ p_1^a + ip_2^a & p_0^a - p_3^a \end{array} \right)$$
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\]

Useful Lorentz-invariant scalars:
\[
\langle ab \rangle \equiv \begin{vmatrix} \lambda^1_a & \lambda^1_b \\ \lambda^2_a & \lambda^2_b \end{vmatrix} , \quad [ab] \equiv \begin{vmatrix} \tilde{\lambda}^1_a & \tilde{\lambda}^1_b \\ \tilde{\lambda}^2_a & \tilde{\lambda}^2_b \end{vmatrix}
\]

\[
(p_a + p_b)^2 = \langle ab \rangle [ba] \equiv s_{ab} , \quad \langle a| (b+\ldots+c)|d \rangle \equiv \langle a| (b)[b+\ldots+c][c]|d \rangle.
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Simple Sources of Simplification: $\mathcal{N} = 4$ Supersymmetry

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In $\mathcal{N} = 4$, all external states are related by supersymmetry.

- at tree-level, pure-glue amplitudes are the same in $\mathcal{N} = 4$ and $\mathcal{N} = 0$
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$N^k$ MHV Classification of Amplitudes

- $A_n^{(m=0)} (+, \ldots, +) = 0$
- $A_n^{(1)} (+, \ldots, - , \ldots, +) = 0 \ (n > 3)$
- $A_n^{(2)} (j^- , \ldots, k^-) = \frac{\langle j \ k \rangle^4}{\langle 1 \ 2 \rangle \langle 2 \ 3 \rangle \cdots \langle n \ 1 \rangle}$
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\[ A_n^{(1)} (+, \ldots, -, \ldots, +) = 0 \quad (n > 3) \]
\[ A_n^{(2)} (j^-, \ldots, k^-) = \frac{\langle j k \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n 1 \rangle} \]
An $n$-point scattering amplitude is specified by listing each particle’s:

- momentum, (which we take to be incoming)
- helicity
- colour

In $\mathcal{N} = 4$, all external states are related by supersymmetry.

- at tree-level, pure-glue amplitudes are the same in $\mathcal{N} = 4$ and $\mathcal{N} = 0$
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### $N^k$MHV Classification of Amplitudes

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- $A_n^{(1)}(+,\ldots,−,\ldots,+)=0$ $(n>3)$
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Simple Sources of Simplification: $\mathcal{N} = 4$ Supersymmetry

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Preliminaries: The (Tree-Level) Analytic S-Matrix, Redux
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Colour & Kinematics: the Vernacular of the S-Matrix
Tree-Level Recursion: Making the Impossible, Possible
Momentum Twistors and Geometry: Trivializing Kinematics

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Tree amplitudes are entirely fixed by analyticity.
Consider the simplest deformation of any amplitude: \( A_n \mapsto \hat{A}_n(z) \)

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![Diagram](image-url)
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\[
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\]

\[
\lambda_1, \tilde{\lambda}_n
\]

\( \lambda_1, \tilde{\lambda}_n \)

The All-Loop S-Matrix of \( \mathcal{N} = 4 \) Super Yang-Mills
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20th January 2011 University of North Carolina at Chapel Hill

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$$
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Analytic S-Matrix Redux: Tree-Level Recursion Relations

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\[ \begin{array}{c}
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\end{array} \]
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$$= \sum \frac{j}{(p_1 + \ldots + p_j)^2}$$

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When the Impossible Becomes Possible

The BCFW tree-level recursion relations made it extremely simple to generate theoretical ‘data’ about scattering amplitudes.

- Amplitudes are calculated with maximum efficiency
- Every term has an interpretation as a leading singularity
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\[ A_6^{(3)}(+) = \frac{1 + g^2 + g^4}{s_{561} \langle 6 \mid 1 \rangle \langle 5 \mid 4 \rangle \langle 3 \mid 3 \rangle \langle 1 \mid 1 \rangle \langle 2 \mid 2 \rangle \langle 3 \mid 3 \rangle} \]

\[ = (1 + g^2 + g^4) \frac{s_{561} \langle 5 \mid 6 \rangle \langle 6 \mid 1 \rangle \langle 2 \mid 3 \rangle \langle 3 \mid 4 \rangle \langle 1 \mid 1 \rangle \langle 6 \mid 5 \rangle \langle 5 \mid 4 \rangle \langle 5 \mid 6 \rangle \langle 1 \mid 2 \rangle \langle 1 \mid 1 \rangle \langle 6 \mid 5 \rangle \langle 5 \mid 4 \rangle}{s_{561} \langle 5 \mid 6 \rangle \langle 6 \mid 1 \rangle \langle 2 \mid 3 \rangle \langle 3 \mid 4 \rangle \langle 1 \mid 1 \rangle \langle 6 \mid 5 \rangle \langle 5 \mid 4 \rangle \langle 5 \mid 6 \rangle \langle 1 \mid 2 \rangle \langle 1 \mid 1 \rangle \langle 6 \mid 5 \rangle \langle 5 \mid 4 \rangle} \]

\[ \langle 6 \mid (2 + 3 + 4) \rangle \langle 1 \mid (6 + 5) \rangle \langle 5 \mid (6 + 1) \rangle \]

\[ e.g. \text{the alternating 6-point NMHV amplitude can be written:} \]

\[ A_6^{(3)}(+) = (1 + g^2 + g^4) \frac{s_{561} \langle 5 \mid 6 \rangle \langle 6 \mid 1 \rangle \langle 2 \mid 3 \rangle \langle 3 \mid 4 \rangle \langle 1 \mid 1 \rangle \langle 6 \mid 5 \rangle \langle 5 \mid 4 \rangle \langle 5 \mid 6 \rangle \langle 1 \mid 2 \rangle \langle 1 \mid 1 \rangle \langle 6 \mid 5 \rangle \langle 5 \mid 4 \rangle}{s_{561} \langle 5 \mid 6 \rangle \langle 6 \mid 1 \rangle \langle 2 \mid 3 \rangle \langle 3 \mid 4 \rangle \langle 1 \mid 1 \rangle \langle 6 \mid 5 \rangle \langle 5 \mid 4 \rangle \langle 5 \mid 6 \rangle \langle 1 \mid 2 \rangle \langle 1 \mid 1 \rangle \langle 6 \mid 5 \rangle \langle 5 \mid 4 \rangle} \]
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but it can **also** be written:

\[
A^{(3)}_6(+, -, +, -, +, -) = (1 + g^2 + g^4) \frac{\langle 4 6 \rangle^4 \langle 1 3 \rangle^4}{s_{456} \langle 4 5 \rangle \langle 5 6 \rangle [1 2] [2 3] \langle 4 | (5 + 6) | 1 \rangle \langle 6 | (5 + 4) | 3 \rangle}
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For 8-point $N^2$MHV, there are 74 linearly-independent 40-term identities connecting the different BCFW formulae.
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\[ A_6^{(3)}(+, -, +, -, +, -) = (1+g^2+g^4) \frac{s_{561}\langle 5\,6\rangle \langle 6\,1 \rangle \langle 2\,3 \rangle \langle 3\,4 \rangle \langle 1|(6+5)|4\rangle \langle 5|(6+1)|2]}{[2\,3]\,[3\,4]} \]

![Diagram of scattering amplitudes](image-url)
When the Impossible Becomes Possible

The BCFW tree-level recursion relations made it extremely simple to generate theoretical ‘data’ about scattering amplitudes.

- Amplitudes are calculated with maximum efficiency
  - but with enormous flexibility
- Every term has an interpretation as a leading singularity
  - but with even more flexibility
- Each term manifests all the symmetries of the theory
  - including those only recently discovered

*E.g.*, the alternating 6-point NMHV amplitude can be written:

\[
A_6^{(3)}(+, -, +, -, +, -) = (1+g^2+g^4) \frac{s_{561} \langle 6 \mid (2 + 3 + 4) \mid 3 \rangle^4}{\langle 6 \mid (5 6) \langle 6 \mid 1 \rangle \langle 2 \mid 3 \rangle [3 \mid 4 \rangle \langle 1 \mid (5 + 6) \rangle \langle 4 \mid (5 + 1) \rangle \langle 2 \mid}
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Solution: *dual-coordinate $x$-space.*

- $p_a \equiv x_{a+1} - x_a$
- scattering amplitudes turn out to be superconformal invariant with respect to these dual-coordinates!
- combined with the ordinary-space superconformal invariance, scattering amplitudes are invariant under an infinite-dimensional Yangian symmetry.
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- $\langle a \ b \ c \ d \rangle \equiv \det (Z_a \ Z_b \ Z_c \ Z_d) = 0 \iff$ the twistors $Z_a, Z_b, Z_c, Z_d$ are linearly dependent.
- So, $(p_a + \ldots + p_b)^2 = 0 \iff \langle a-1 a b b+1 \rangle = 0$. 

![Hexagon diagram with momenta and variables]
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\mathcal{A}_n^{(m)} = \sum_{\text{partitions of } n,m} \mathcal{A}_{n_L}^{(m_L)}(1, \ldots, j, \hat{J}) \otimes \mathcal{A}_{n_R}^{(m_R)}(\hat{J}, j+1, \ldots, n-1, \hat{n})
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\]

\( \hat{J} \equiv (j, j+1) \cap (n-1 \, n \, 1) \) and \( \hat{n} \equiv (n, n-1) \cap (j, j+1, 1) \)

\[\begin{align*}
1 & \quad n \\
\sum_{j} & \quad \bigotimes BCFW
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\[
\begin{array}{c}
\text{n - 1} \\
\bullet \\
\text{n} \\
\bullet \\
\text{j + 1} \\
\bullet \\
\text{1} \\
\bullet \\
\text{j} \\
\bullet
\end{array}
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**Tree-Level BCFW in Momentum-Twistor Variables**
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$$A_n^{(m)} = \sum_{\text{partitions of } n,m} A_{nL}^{(mL)}(1, \ldots, j, \hat{J}) \otimes BCFW A_{nR}^{(mR)}(\hat{J}, j+1, \ldots, n-1, \hat{n})$$

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$$\mathcal{A}^{(m)}_n = \sum_{\text{partitions of } n,m} \mathcal{A}^{(m_L)}_{n_L}(1, \ldots, j, \hat{J}) \bigotimes \mathcal{A}^{(m_R)}_{n_R}(\hat{J}, j+1, \ldots, n-1, \hat{n})$$

$\hat{J} \equiv (j j+1) \cap (n-1 n 1)$ and $\hat{n} \equiv (n n-1) \cap (j j+1 1)$
Because in momentum-twistor variables momentum conservation is automatic, the ‘naïve’ analytic continuation works:

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The Most Useful Identity in Projective Geometry:

\[ Z_a\langle b \, c \, d \, e \rangle + Z_b\langle c \, d \, e \, a \rangle + Z_c\langle d \, e \, a \, b \rangle + Z_d\langle e \, a \, b \, c \rangle + Z_e\langle a \, b \, c \, d \rangle = 0. \]
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The Most Useful Identity in Projective Geometry:

$$-Z_{a}\langle b \ c \ d \ e \rangle = Z_{b}\langle c \ d \ e \ a \rangle + Z_{c}\langle d \ e \ a \ b \rangle + Z_{d}\langle e \ a \ b \ c \rangle + Z_{e}\langle a \ b \ c \ d \rangle$$
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**The Most Useful Identity in Projective Geometry:**

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-Z_a\langle b c d e \rangle - Z_b\langle c d e a \rangle = Z_c\langle d e a b \rangle + Z_d\langle e a b c \rangle + Z_e\langle a b c d \rangle
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$$\hat{J} \equiv (j \, j+1) \cap (n-1 \, n \, 1) = Z_j \langle j+1 \, n-1 \, n \, 1 \rangle + Z_{j+1} \langle n-1 \, n \, 1 \, j \rangle$$
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The Meaning of *The* Loop Integrand

In a general theory, there is no naturally well-defined way to combine disparate Feynman loop integrals:

\[
\begin{align*}
&\quad = \left\{ \int d^4 \ell_1 \frac{(p_1 + p_2)^2(p_2 + p_3)^2}{\ell_1^2(\ell_1 - p_1)^2(\ell_1 - p_1 - p_2)^2(\ell_1 + p_4)^2}, \\
&\quad \int d^4 \ell_2 \frac{(p_1 + p_2)^2(p_2 + p_3)^2}{\ell_2^2(\ell_2 - p_2)^2(\ell_2 - p_1 - p_2)^2(\ell_2 + p_4)^2} \right\}
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At least for planar theories, the loop-integrand is unambiguous.

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L &= \int d^4 L rac{(p_1 + p_2)^2 (p_2 + p_3)^2}{L^2 (L - p_1)^2 (L - p_1 - p_2)^2 (L + p_4)^2}
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\]

In dual coordinates, we find

\[
= \int d^4 x \frac{(x_1 - x_3)^2(x_2 - x_4)^2}{(x - x_1)^2(x - x_2)^2(x - x_3)^2(x - x_4)^2}
\]
Integrals over Lines in Momentum-Twistor Space

Integration over all $x$ corresponds to the integration over all lines $(Z_A Z_B)$ in momentum-twistor space.

$$\int d^4 x \iff \int \frac{d^4 Z_A d^4 Z_B}{\text{vol}(GL_2) \times \langle \lambda_A \lambda_B \rangle^4} \equiv \int_{AB}$$

The propagators are

$$(x - x_1)^2 \iff \langle AB \ 12 \rangle \quad (x - x_2)^2 \iff \langle AB \ 23 \rangle \quad \text{etc.}$$

and the integral becomes

$$\int_{AB} \frac{\langle 12 \ 34 \rangle^2}{\langle AB \ 12 \rangle \langle AB \ 23 \rangle \langle AB \ 34 \rangle \langle AB \ 41 \rangle}$$
The Origin of Loop Amplitudes: Forward Limits

Let us reconsider the BCFW deformation for momentum-twistors:

\[ Z_n \mapsto Z_n + z Z_{n-1}. \]

- The ordinary terms come from factorizations: \( \langle \hat{n} 1 j j+1 \rangle = 0. \)
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\]

\[
= \sum_j \text{BCFW}
\]

The All-Loop S-Matrix of \( \mathcal{N} = 4 \) Super Yang-Mills
In $\mathcal{N} = 4$ these forward limits are always well-defined and finite

- the same has been proven for up to two-loops in any supersymmetric theory

There is evidence that there exists a ‘smart forward limit’ that is always finite and well-defined in any planar theory, extending the all-loop recursion to even pure-glue (in the planar limit).
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Caron – Huot
arXiv:1007.3224
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arXiv:1007.3224
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Exempli Gratia: BCFW Form of MHV Loop Amplitudes

Taking the forward limit of an \((n+2)\)-point NMHV tree amplitude we find the following expression for the one-loop MHV amplitude:

\[
\sum_{i<j} \int \frac{\langle AB (1 i i+1) \cap (1 j j+1) \rangle}{\langle AB 1 i \rangle \langle AB i i+1 \rangle \langle AB i+1 1 \rangle \langle AB 1 j \rangle \langle AB j j+1 \rangle \langle AB j+1 1 \rangle}
\]
Sewing Together Tree Amplitudes in $\mathcal{N} = 4$
Sewing Together Tree Amplitudes in $\mathcal{N} = 4$

Two-Mass-Easy Schubert Problem
Sewing Together Tree Amplitudes in $\mathcal{N} = 4$

Two-Mass-Easy Schubert Problem

\[
\begin{array}{c}
\begin{array}{c}
2 \quad 3 \\
\text{ } \\\\downarrow \\
4 \\
\text{ } \\\\uparrow \\
1 \quad 5 \\
\end{array}
\end{array}
\rightleftharpoons
\int_{AB} \frac{\langle 123 \ 5 \rangle \langle 2 \ 345 \rangle}{\langle AB \ 12 \rangle \langle AB \ 23 \rangle \langle AB \ 45 \rangle \langle AB \ 56 \rangle},
\]

\[
\begin{array}{c}
\begin{array}{c}
\mathcal{A}_1^{m_1} \\
\text{ } \\\\downarrow \\
\mathcal{A}_2^{m_2} \\
\text{ } \\\\uparrow \\
\mathcal{A}_4^{m_4} \\
\text{ } \\\\downarrow \\
\mathcal{A}_3^{m_3} \\
\end{array}
\end{array}
\]
Sewing Together Tree Amplitudes in $\mathcal{N} = 4$

Two-Mass-Easy Schubert Problem

$$
\frac{\langle 123 \, 5 \rangle \langle 2 \, 345 \rangle}{\langle AB \, 12 \rangle \langle AB \, 23 \rangle \langle AB \, 45 \rangle \langle AB \, 56 \rangle}
$$

$$(AB) = (25)$$

$$(AB) = (123) \cap (456)$$
Finite Integrals in Momentum Twistor Space

\[ \int_{A,B} \frac{\langle AB(j-1 \ j \ j+1) \cap (k-1 \ k \ k+1) \rangle \langle 1 \ 2 \ j \ k \rangle}{\langle AB \ 12 \rangle \langle AB \ j-1 \ j \rangle \langle AB \ j \ j+1 \rangle \langle AB \ k-1 \ k \rangle \langle AB \ k \ k+1 \rangle} \]

The All-Loop S-Matrix of $\mathcal{N} = 4$ Super Yang-Mills
Finite Integrals in Momentum Twistor Space

\[ \int \frac{\langle AB(j-1,j,j+1) \cap (k-1,k,k+1) \rangle \langle 1,2,j,k \rangle}{\langle AB,1,2 \rangle \langle AB,j-1,j \rangle \langle AB,j,j+1 \rangle \langle AB,k-1,k \rangle \langle AB,k,k+1 \rangle} = \text{Li}_2(1 - u_1) \]

\[ u_1 \equiv \frac{\langle k,k+1,1,2 \rangle \langle j-1,j,k-1,k \rangle}{\langle k,k+1,j-1,j \rangle \langle 1,2,k-1,k \rangle} \]
Finite Integrals in Momentum Twistor Space

\[
\int_{AB} \frac{\langle AB(j-1 \, j \, j+1) \cap (k-1 \, k \, k+1) \rangle \langle 1 \, 2 \, j \, k \rangle}{\langle AB \, 1 \, 2 \rangle \langle AB \, j-1 \, j \rangle \langle AB \, j \, j+1 \rangle \langle AB \, k-1 \, k \rangle \langle AB \, k \, k+1 \rangle} = \text{Li}_2(1 - u_1) + \text{Li}_2(1 - u_2)
\]

\[
u_1 \equiv \frac{\langle k \, k+1 \, 1 \, 2 \rangle \langle j-1 \, j \, k-1 \, k \rangle}{\langle k \, k+1 \, j-1 \, j \rangle \langle 1 \, 2 \, k-1 \, k \rangle}
\]

\[
u_2 \equiv \frac{\langle j \, j+1 \, k \, k+1 \rangle \langle 1 \, 2 \, j-1 \, j \rangle}{\langle j \, j+1 \, 1 \, 2 \rangle \langle k \, k+1 \, j-1 \, j \rangle}
\]
Finite Integrals in Momentum Twistor Space

\[
\int_{AB} \frac{\langle AB(j-1\ j\ j+1) \cap (k-1\ k\ k+1) \rangle \langle 1\ 2\ j\ k \rangle}{\langle AB\ 12 \rangle \langle AB\ j-1\ j \rangle \langle AB\ j\ j+1 \rangle \langle AB\ k-1\ k \rangle \langle AB\ k\ k+1 \rangle} = \text{Li}_2(1 - u_1) + \text{Li}_2(1 - u_2) - \text{Li}_2(1 - u_3)
\]

\[u_1 \equiv \frac{\langle k\ k+1\ 1\ 2 \rangle \langle j-1\ j\ k-1\ k \rangle}{\langle k\ k+1\ j-1\ j \rangle \langle 1\ 2\ k-1\ k \rangle}\]

\[u_2 \equiv \frac{\langle j\ j+1\ k\ k+1 \rangle \langle 1\ 2\ j-1\ j \rangle}{\langle j\ j+1\ 1\ 2 \rangle \langle k\ k+1\ j-1\ j \rangle}\]

\[u_3 \equiv \frac{\langle k\ k+1\ 1\ 2 \rangle \langle j\ j+1\ k\ k-1\ k \rangle}{\langle k\ k+1\ j\ j+1 \rangle \langle 1\ 2\ k-1\ k \rangle}\]
Finite Integrals in Momentum Twistor Space

\[ \int_{AB} \frac{\langle AB(j−1 j j+1) \cap (k−1 k k+1) \rangle \langle 1 2 j k \rangle}{\langle AB 12 \rangle \langle AB j−1 j \rangle \langle AB j j+1 \rangle \langle AB k−1 k \rangle \langle AB k k+1 \rangle} = \text{Li}_2(1 − u_1) + \text{Li}_2(1 − u_2) − \text{Li}_2(1 − u_3) − \text{Li}_2(1 − u_4) \]

\[ u_1 \equiv \frac{\langle k k+1 1 2 \rangle \langle j−1 j k−1 k \rangle}{\langle k k+1 j−1 j \rangle \langle 1 2 k−1 k \rangle} \]

\[ u_2 \equiv \frac{\langle j j+1 k k+1 \rangle \langle 1 2 j−1 j \rangle}{\langle j j+1 1 2 \rangle \langle k k+1 j−1 j \rangle} \]

\[ u_3 \equiv \frac{\langle k k+1 1 2 \rangle \langle j j+1 k−1 k \rangle}{\langle k k+1 j j+1 \rangle \langle 1 2 k−1 k \rangle} \]

\[ u_4 \equiv \frac{\langle j j+1 k−1 k \rangle \langle 1 2 j−1 j \rangle}{\langle j j+1 1 2 \rangle \langle k−1 k j−1 j \rangle} \]
Finite Integrals in Momentum Twistor Space

\[ \int_{AB} \frac{\langle AB(j-1 \ j \ j+1) \cap (k-1 \ k \ k+1) \rangle \langle 1 \ 2 \ j \ k \rangle}{\langle AB \ 12 \rangle \langle AB \ j-1 \ j \rangle \langle AB \ j \ j+1 \rangle \langle AB \ k-1 \ k \rangle \langle AB \ k \ k+1 \rangle} = \text{Li}_2(1 - u_1) + \text{Li}_2(1 - u_2) - \text{Li}_2(1 - u_3) - \text{Li}_2(1 - u_4) + \text{Li}_2(1 - u_5) \]

\[ u_1 \equiv \frac{\langle k \ k+1 \ 1 \ 2 \rangle \langle j-1 \ j \ k-1 \ k \rangle}{\langle k \ k+1 \ j-1 \ j \rangle \langle 1 \ 2 \ k-1 \ k \rangle} \]

\[ u_2 \equiv \frac{\langle j \ j+1 \ k \ k+1 \rangle \langle 1 \ 2 \ j-1 \ j \rangle}{\langle j \ j+1 \ 1 \ 2 \rangle \langle k \ k+1 \ j-1 \ j \rangle} \]

\[ u_3 \equiv \frac{\langle k \ k+1 \ 1 \ 2 \rangle \langle j \ j+1 \ k-1 \ k \rangle}{\langle k \ k+1 \ j \ j+1 \rangle \langle 1 \ 2 \ k-1 \ k \rangle} \]

\[ u_4 \equiv \frac{\langle j \ j+1 \ k-1 \ k \rangle \langle 1 \ 2 \ j-1 \ j \rangle}{\langle j \ j+1 \ 1 \ 2 \rangle \langle k-1 \ k \ j-1 \ j \rangle} \]

\[ u_5 \equiv \frac{\langle j \ j+1 \ k-1 \ k \rangle \langle k \ k+1 \ j-1 \ j \rangle}{\langle j \ j+1 \ k \ k+1 \rangle \langle 1 \ 2 \ k-1 \ j \rangle} \]

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The All-Loop S-Matrix of $\mathcal{N} = 4$ Super Yang-Mills
Finite Integrals in Momentum Twistor Space

\[
\int AB_{12} \left( \frac{\langle AB(j-1 \ j \ j+1) \cap (k-1 \ k \ k+1) \rangle \langle 1 \ 2 \ j \ k \rangle}{\langle AB \ 12 \rangle \langle AB \ j-1 \ j \rangle \langle AB \ j \ j+1 \rangle \langle AB \ k-1 \ k \rangle \langle AB \ k \ k+1 \rangle} \right)
\]

\[= \text{Li}_2(1 - u_1) + \text{Li}_2(1 - u_2) - \text{Li}_2(1 - u_3) - \text{Li}_2(1 - u_4) + \text{Li}_2(1 - u_5) + \log(u_1) \log(u_2)
\]

\[u_1 \equiv \frac{\langle k \ k+1 \ 1 \ 2 \rangle \langle j-1 \ j \ k-1 \ k \rangle}{\langle k \ k+1 \ j-1 \ j \rangle \langle 1 \ 2 \ k-1 \ k \rangle}
\]

\[u_2 \equiv \frac{\langle j \ j+1 \ k \ k+1 \rangle \langle 1 \ 2 \ j-1 \ j \rangle}{\langle j \ j+1 \ 1 \ 2 \rangle \langle k \ k+1 \ j-1 \ j \rangle}
\]

\[u_3 \equiv \frac{\langle k \ k+1 \ 1 \ 2 \rangle \langle j \ j+1 \ k-1 \ k \rangle}{\langle k \ k+1 \ j \ j+1 \rangle \langle 1 \ 2 \ k-1 \ k \rangle}
\]

\[u_4 \equiv \frac{\langle j \ j+1 \ k-1 \ k \rangle \langle 1 \ 2 \ j-1 \ j \rangle}{\langle j \ j+1 \ 1 \ 2 \rangle \langle k-1 \ k \ j-1 \ j \rangle}
\]

\[u_5 \equiv \frac{\langle j \ j+1 \ k-1 \ k \rangle \langle k \ k+1 \ j-1 \ j \rangle}{\langle j \ j+1 \ k \ k+1 \rangle \langle k-1 \ k \ j-1 \ j \rangle}
\]
Finite Integrals in Momentum Twistor Space

\[ \int_{AB} \frac{\langle AB(j-1\ j\ j+1) \cap (k-1\ k\ k+1) \rangle \langle 1\ 2\ j\ k \rangle}{\langle AB\ 1\ 2 \rangle \langle AB\ j-1\ j \rangle \langle AB\ j\ j+1 \rangle \langle AB\ k-1\ k \rangle \langle AB\ k\ k+1 \rangle} \]

\[ = \text{Li}_2(1 - u_1) + \text{Li}_2(1 - u_2) - \text{Li}_2(1 - u_3) - \text{Li}_2(1 - u_4) + \text{Li}_2(1 - u_5) + \log(u_1) \log(u_2) \]

\[ u_2 \equiv \frac{\langle j\ j+1\ k\ k+1 \rangle \langle 1\ 2\ j-1\ j \rangle}{\langle j\ j+1\ 1\ 2 \rangle \langle k\ k+1\ j-1\ j \rangle} \]

\[ u_3 \equiv \frac{\langle k\ k+1\ 1\ 2 \rangle \langle j\ j+1\ k-1\ k \rangle}{\langle k\ k+1\ j\ j+1 \rangle \langle 1\ 2\ k-1\ k \rangle} \]

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Finite Integrals in Momentum Twistor Space

\[
\int_{\mathcal{A}\mathcal{B}} \frac{\langle \mathcal{A}\mathcal{B}(j-1 \ j \ j+1) \cap (k-1 \ k \ k+1) \rangle \langle 1 \ 2 \ j \ k \rangle}{\langle \mathcal{A}\mathcal{B} \ 1\!2 \rangle \langle \mathcal{A}\mathcal{B} \ j-1 \ j \rangle \langle \mathcal{A}\mathcal{B} \ j \ j+1 \rangle \langle \mathcal{A}\mathcal{B} \ k-1 \ k \rangle \langle \mathcal{A}\mathcal{B} \ k \ k+1 \rangle} = \text{Li}_2(1 - u_1) + \text{Li}_2(1 - u_2) - \text{Li}_2(1 - u_3) - \text{Li}_2(1 - u_4) + \text{Li}_2(1 - u_5) + \log(u_1) \log(u_2)
\]

\[
u_5 \equiv \frac{\langle j \ j+1 \ k-1 \ k \rangle \langle k \ k+1 \ j-1 \ j \rangle}{\langle j \ j+1 \ k \ k+1 \rangle \langle k-1 \ k \ j-1 \ j \rangle}
\]

\[
u_3 \equiv \frac{\langle k \ k+1 \ 1 \ 2 \rangle \langle j \ j+1 \ k-1 \ k \rangle}{\langle k \ k+1 \ j \ j+1 \rangle \langle 1 \ 2 \ k-1 \ k \rangle}
\]

\[
u_4 \equiv \frac{\langle j \ j+1 \ k-1 \ k \rangle \langle 1 \ 2 \ j-1 \ j \rangle}{\langle j \ j+1 \ 1 \ 2 \rangle \langle k-1 \ k \ j-1 \ j \rangle}
\]
Finite Integrals in Momentum Twistor Space

\[ \int_{AB} \frac{\langle AB(j-1 \ j \ j+1) \cap (k-1 \ k \ k+1) \rangle \langle 1 \ 2 \ j \ k \rangle}{\langle AB \ 12 \rangle \langle AB \ j-1 \ j \rangle \langle AB \ j \ j+1 \rangle \langle AB \ k-1 \ k \rangle \langle AB \ k \ k+1 \rangle} = Li_2(1 - u_1) + Li_2(1 - u_2) - Li_2(1 - u_3) - Li_2(1 - u_4) + Li_2(1 - u_5) + \log(u_1) \log(u_2) \]

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The All-Loop S-Matrix of \( \mathcal{N} = 4 \) Super Yang-Mills
Finite Integrals in Momentum Twistor Space

\[
\int_{\mathcal{A}\mathcal{B}} \frac{\langle AB\, (j-1 \ j \ j+1) \cap (k-1 \ k \ k+1) \rangle \langle 1 \ 2 \ j \ k \rangle}{\langle AB\, 12 \rangle \langle AB\, j-1 \ j \rangle \langle AB\, j \ j+1 \rangle \langle AB\, k-1 \ k \rangle \langle AB\, k \ k+1 \rangle} = \text{Li}_2(1-u_1) + \text{Li}_2(1-u_2) - \text{Li}_2(1-u_3) - \text{Li}_2(1-u_4) + \text{Li}_2(1-u_5) + \log(u_1)\log(u_2)
\]

\[
u_5 = \frac{\langle j \ j+1 \ k-1 \ k \rangle \langle k \ k+1 \ j-1 \ j \rangle}{\langle j \ j+1 \ k \ k+1 \rangle \langle k-1 \ k \ j-1 \ j \rangle}
\]
Finite Integrals in Momentum Twistor Space

\[ \int_{AB} \frac{\langle AB(j-1 \ j \ j+1) \cap (k-1 \ k \ k+1) \rangle \langle 1 \ 2 \ j \ k \rangle}{\langle AB \ 1 2 \rangle \langle AB \ j-1 \ j \rangle \langle AB \ j \ j+1 \rangle \langle AB \ k-1 \ k \rangle \langle AB \ k \ k+1 \rangle} = \text{Li}_2(1 - u_1) + \text{Li}_2(1 - u_2) - \text{Li}_2(1 - u_3) - \text{Li}_2(1 - u_4) + \text{Li}_2(1 - u_5) + \log(u_1) \log(u_2) \]
Finite Integrals in Momentum Twistor Space

\[
\int \frac{\langle AB(j-1 \ j \ j+1) \cap (k-1 \ k \ k+1) \rangle \langle 1 \ 2 \ j \ k \rangle}{\langle AB \ 12 \rangle \langle AB \ j-1 \ j \rangle \langle AB \ j \ j+1 \rangle \langle AB \ k-1 \ k \rangle \langle AB \ k \ k+1 \rangle} = \text{Li}_2(1 - u_1) + \text{Li}_2(1 - u_2) - \text{Li}_2(1 - u_3) - \text{Li}_2(1 - u_4) + \text{Li}_2(1 - u_5) + \log(u_1) \log(u_2)
\]
Finite Integrals in Momentum Twistor Space

\[
\int_{AB} \frac{\langle AB(j-1 \ j \ j+1) \cap (k-1 \ k \ k+1) \rangle \langle 1 \ 2 \ j \ k \rangle}{\langle AB \ 12 \rangle \langle AB \ j-1 \ j \rangle \langle AB \ j \ j+1 \rangle \langle AB \ k-1 \ k \rangle \langle AB \ k \ k+1 \rangle} = \text{Li}_2(1 - u_1) + \text{Li}_2(1 - u_2) - \text{Li}_2(1 - u_3) - \text{Li}_2(1 - u_4) + \text{Li}_2(1 - u_5) + \log(u_1) \log(u_2)
\]
In recent months, similar simplifications have been ‘guessed’ (and checked):

\[
A_n^{(2)}(\ldots, j^-, \ldots, k^-, \ldots) = \frac{\langle j \, k \rangle^4}{\langle 1 \, 2 \rangle \langle 2 \, 3 \rangle \cdots \langle n \, 1 \rangle}
\]
The Continuation of this Logic Through 3-Loops:

In recent months, similar simplifications have been ‘guessed’ (and checked):

\[ \mathcal{A}^{(2)}_{n}(\ldots, j^{-}, \ldots, k^{-}, \ldots) = \frac{\langle j \ k \rangle^{4}}{\langle 1 \ 2 \rangle \langle 2 \ 3 \rangle \cdots \langle n \ 1 \rangle} \]

\[ \times \left\{ 1 \right\} \]
The Continuation of this Logic Through 3-Loops:

In recent months, similar simplifications have been ‘guessed’ (and checked):

\[ A_{n}^{(2)}(\ldots, j, \ldots, k, \ldots) = \frac{\langle j \, k \rangle^4}{\langle 1 \, 2 \rangle \langle 2 \, 3 \rangle \ldots \langle n \, 1 \rangle} \times \left\{ 1 + \sum_{i < j < i} \right\} \]

\[ \begin{array}{c}
\text{Diagram}
\end{array} \]
In recent months, similar simplifications have been ‘guessed’ (and checked):

\[ A_n^{(2)}(\ldots, j^-, \ldots, k^-, \ldots) = \frac{\langle j \, k \rangle^4}{\langle 1 \, 2 \rangle \langle 2 \, 3 \rangle \cdots \langle n \, 1 \rangle} \]

\[
\times \left\{ 1 + \sum_{i<j<i}^{i<j<i} \right\} + \frac{1}{2} \sum_{i<j<k<l<i}^{i<j<k<l<i}
\]
The Continuation of this Logic Through 3-Loops:

In recent months, similar simplifications have been ‘guessed’ (and checked):

\[
\mathcal{A}^{(2)}_{n}(\ldots, j^{-}, \ldots, k^{-}, \ldots) = \frac{\langle j \, k \rangle^4}{\langle 1 \, 2 \rangle \langle 2 \, 3 \rangle \cdots \langle n \, 1 \rangle}
\]

\[
\times \left\{ 1 + \sum_{i<j<i} X \right\} + \frac{1}{2} \sum_{i<j<k<l<i} j \]

\[
+ \frac{1}{3} \sum_{i_1 \leq i_2 < j_1 \leq j_2 < k_1 \leq k_2 < i_1} \]

\[
+ \frac{1}{2} \sum_{i_1 \leq j_1 < k_1 < k_2 \leq j_2 < i_2 < i_1} \]

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The All-Loop S-Matrix of $\mathcal{N} = 4$ Super Yang-Mills
Forward Looking Comments

- Do there exist alternative, e.g. purely geometric ways of characterizing the full S-Matrix?
- How can we systematically regulate and compute momentum-twistor loop integrals?
  - Can we perform these integrals analytically at the outset?
  - Deeper connections to the leading-singularity programme?
  - Connections to ‘symbols’ & mixed Tate motives?
  - How should the integrals coming from recursions be done directly?
- How easy is it to extend these results to other theories?
  - Non-supersymmetric (planar) Yang-Mills?
  - Non-planar theories?
  - Massive theories?
- ...
Forward Looking Comments

- Do there exist alternative, *e.g.* purely geometric ways of characterizing the full S-Matrix?

- **How can we systematically regulate and compute momentum-twistor loop integrals?**
  - Can we perform these integrals analytically at the outset?
  - Deeper connections to the leading-singularity programme?
  - Connections to ‘symbols’ & mixed Tate motives?
  - How should the integrals coming from recursions be done directly?

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