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Cohomology of the non-Abelian Seiberg-Witten map

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Abstract: Study of the non-Abelian Seiberg-Witten map by a cohomological approach. We introduce ghosts and determine the coboundary operator. This allows us to find solutions of the map by constructing a corresponding homotopy operator and clarifies the nature of the ambiguities which arise.

Solutions of the SW map are also computed by means of a differential equation.

Plan:

- Gauge theory on noncommutative spaces
- Seiberg-Witten map
- Introduction of ghosts and of the coboundary operator
- Construction of the corresponding homotopy operator
- Seiberg-Witten differential equation

Gauge theory on noncommutative spaces

Space-time commutation relations

x^i coordinates, $i = 1, \ldots, D$
D space-time dimension

$$\left[x^i \overset{\star}{,} x^j\right] = i\ \theta^{ij}$$

where θ is the constant Poisson tensor

$$\theta^{ij} = -\theta^{ji}$$

and the Weyl-Moyal product is defined by

$$\begin{aligned} f \star g &= f e^{\frac{i}{2}\theta^{ij}\overleftarrow{\partial_i}\overrightarrow{\partial_j}} g \\ &= fg + \frac{1}{2} i\theta^{ij}\partial_i f \partial_j g \\ &\quad -\frac{1}{8}\theta^{ij} t^{kl}\partial_i\partial_k f\ \partial_j\partial_l g + O[t^3] \end{aligned}$$

with $\partial_i \equiv \frac{\partial}{\partial x^i}$

It is an associative but not commutative product.

$$(f \star g) \star h = f \star (g \star h)$$

Relation with string theory

In string theory the Poisson tensor θ^{ij} is related to the antisymmetric tensor B^{ij} by the formula

$$\theta^{ij} = 2\pi\alpha' \left(\frac{1}{g + 2\pi\alpha' B}\right)^{[ij]}$$

where [ ] antisymmetric part, g metric, α' string tension

In principle θ should be treated as a dynamical field and is therefore not necessarily constant. But we restrict ourselves to the case of constant θ.

In the limit $\alpha' B >> g$ we have the simple relation

$$\theta^{ij} = \frac{1}{B_{ij}}$$

Seiberg-Witten map

Seiberg and Witten, JHEP09(1999)032

Gauge transformation on commutative space
a_i gauge potential, α gauge parameter

$$\delta_\alpha a_i = \partial_i \alpha - i[a_i, \alpha]$$

Gauge transformation on noncommutative space
A_i gauge potential, Λ gauge parameter

$$A_i = A_i(\, a, \partial a, \partial^2 a, \cdots)$$

$$\Lambda = \Lambda(\, \alpha, \partial \alpha, \cdots, a, \partial a, \cdots),$$

$$\delta_\Lambda A_i = \partial_i \Lambda - i[A_i \stackrel{\star}{,} \Lambda] \equiv \partial_i \Lambda - i\,(A_i \star \Lambda - \Lambda \star A_i)$$

Seiberg-Witten equation

We require

$$A_i + \delta_\Lambda A_i = A_i(a_j + \delta_\alpha a_j, \cdots)$$

It is at the same time an equation for both A_i and Λ

Properties of the SW map

- It expresses the non-commutative gauge field and parameter in terms of the commutative ones.

- Usually the algebra of the gauge fields does not close in the noncommutative case and an infinite number of fields is expected. The SW map allows us to express the non-commutative fields in terms of the commutative ones, which are a finite number.

- In string theory the existence of the SW map follows from the fact that two different regularization techniques (Pauli-Villars and point-splitting) lead either to a commutative or a noncommutative theory and therefore the two theories are supposed to be physically equivalent.

- An interaction which is complicated when expressed in terms of the commutative variables becomes a simple free theory in the noncommutative coordinates. The interaction is encoded in the noncommutative structure of the space.

- There are different types of ambiguities in the solutions of the Seiberg-Witten equation, as a consequence of field redefinitions and the dependence on the choice of the path in θ-space.
  (Asakawa and Kishimoto, hep-th/9909139)

Seiberg-Witten equation as a consistency condition

Jurco, Möller, Schraml, Schupp, Wess, hep-th/0104153

Introduce ψ gauge field
On commutative space

$$\delta_\alpha \psi = i\alpha\psi$$

Composition property of gauge transformations

$$\left[\delta_\alpha, \delta_\beta\right]\psi = [\alpha, \beta]\Psi = -i\delta_{[\alpha,\beta]}\psi$$

On noncommuative space

$$\begin{aligned} \Psi &= \Psi(\psi, a, \partial a, \partial^2 a, \ldots) \\ \delta_\Lambda \Psi &= i\Lambda \star \Psi \end{aligned}$$

Require

$$\begin{aligned} \delta_{\Lambda_\alpha}\Psi &= \delta_\alpha \Psi \\ \left[\delta_{\Lambda_\alpha}, \delta_{\Lambda_\beta}\right]\Psi &= \left[\delta_\alpha, \delta_\beta\right]\Psi \end{aligned}$$

From

$$\begin{aligned}\Lambda_{[\alpha,\beta]} &= \left[\delta_{\Lambda_\alpha}, \delta_{\Lambda_\beta}\right] \Psi \\ &= i\left(\delta_\alpha \Lambda_\beta - \delta_\beta \Lambda_\alpha\right) \star \Psi + \left[\Lambda_\alpha \stackrel{\star}{,} \Lambda_\beta\right] \star \Psi\end{aligned}$$

by dropping Ψ follows

$$\left(\delta_\alpha \Lambda_\beta - \delta_\beta \Lambda_\alpha\right) - i\left[\Lambda_\alpha \stackrel{\star}{,} \Lambda_\beta\right] + i\Lambda_{[\alpha,\beta]} = 0$$

Advantages of this formulation:

- The equation for Λ is decoupled from the equation for A_i.
- In the noncommutative case the gauge parameters are elements of the envelopping algebra of the Lie algebra and not necessarily of the Lie algebra, unless we are considering the case of the fundamental representation of $U(n)$. The Seiberg-Witten map allows us to express this infinite number of noncommutative fields in terms of a finite number of commutative fields.

Introduction of the ghost fields and of the coboundary operator

Instead of the gauge parameter α_i use an odd ghost field v and define

$$\begin{aligned} \delta_v v &= i v^2 \\ \delta_v a_i &= \partial_i v - i[a_i, v] \equiv D_i v \end{aligned}$$

with the properties

$$\begin{aligned} \delta_v^2 &= 0 \\ [\delta_v, \partial_i] &= 0 \\ \delta_v(f_1 f_2) &= (\delta_v f_1) f_2 + (-1)^{deg(f_1)} f_1 (\delta_v f_2) \end{aligned}$$

Moreover, define the coboundary operator

$$\Delta = \begin{cases} \delta_v - i\{v, \cdot\} & \text{on odd quantities} \\ \delta_v - i[v, \cdot] & \text{on even quantities} \end{cases}$$

so that

$$\begin{aligned} \Delta v &= -i v^2, \qquad \Delta a_i = \partial_i v \\ \Delta^2 &= 0 \\ [\Delta, D_i] &= 0 \\ \Delta(f_1 f_2) &= (\Delta f_1) f_2 + (-1)^{deg(f_1)} f_1 (\Delta f_2) \end{aligned}$$

Seiberg-Witten equation in the ghost formalism

$$
\begin{aligned}
\delta\Psi &= i\Lambda\star\Psi \\
\delta\Lambda &= i\Lambda\star\Lambda \\
\delta A_i &= \partial_i\Lambda - i[A_i \stackrel{\star}{;} \Lambda]
\end{aligned}
$$

The equation for Λ follows from the nilpotency of δ and the associativity of the star product.

Expansion in θ^{ij}

Gauge parameter

$$
\Lambda = \Lambda^{(0)} + \Lambda^{(1)} + \ldots,
$$

$$
\Lambda^{(0)} = v, \qquad \Lambda^{(1)} = \frac{1}{4}\theta^{ij}\left\{\partial_i v, a_j\right\}
$$

Gauge potential

$$
A_i = A_i^{(0)} + A_i^{(1)} + \ldots
$$

$$
A_i^{(0)} = a_i, \qquad A_i^{(1)} = -\frac{1}{4}\theta^{kl}\left\{a_k, \partial_l a_i + F_{li}\right\}
$$

with $F_{ij} = \partial_i a_j - \partial_j a_i - i[a_i, a_j]$ field strength

Seiberg-Witten equation for Λ:

$$
\begin{aligned}
0^{th}: &\quad \delta_v v = i v^2 \\
1^{st}: &\quad \Delta\Lambda^{(1)} = -\frac{1}{2}\theta^{ij} b_i b_j \\
2^{nd}: &\quad \Delta\Lambda^{(2)} = -\frac{i}{8}\theta^{ij}\theta^{kl}\partial_i b_k \partial_j b_l \\
&\quad -\frac{1}{2}\theta^{ij}[b_i, \partial_j \Lambda^{(1)}] + i\Lambda^{(1)}\Lambda^{(1)}
\end{aligned}
$$

Seiberg-Witten equation for A_i:

$$
\begin{aligned}
0^{th}: &\quad \Delta A_i^{(0)} = b_i \\
1^{st}: &\quad \Delta A_i^{(1)} = D_i\Lambda^{(1)} - \frac{1}{2}\theta^{kl}\{b_k, \partial_l a_i\} \\
2^{nd}: &\quad \Delta A_i^{(2)} = D_i\Lambda^{(2)} + i[\Lambda^{(1)}, A_i^{(1)}] \\
&\quad -\frac{1}{2}\theta^{kl}\{b_k, \partial_l A_i^{(1)}\} - \frac{1}{2}\theta^{kl}\{\partial_k\Lambda^{(1)}, \partial_l a_i\} \\
&\quad -\frac{i}{8}\theta^{kl}\theta^{mn}[\partial_k b_m, \partial_l \partial_n a_i]
\end{aligned}
$$

Here we have introduced the useful notation

$$b_i \equiv \partial_i v$$

General structure of the Seiberg-Witten equation to order n :

$$\begin{aligned} \Delta \Lambda^{(n)} &= M^{(n)}, \\ \Delta A_i^{(n)} &= U_i^{(n)} \end{aligned}$$

Consistency conditions following from $\Delta^2 = 0$:

$$\Delta M^{(n)} = 0, \quad \Delta U_i^{(n)} = 0$$

Ambiguities in the solutions

If Λ and A_i are solutions so are

$$\begin{aligned} \tilde{\Lambda}^{(n)} &= \Lambda^{(n)} + \Delta S^{(n)} \\ \tilde{A}_i^{(n)} &= A_i^{(n)} + D_i S^{(n)} + S_i'^{(n)} \end{aligned}$$

for arbitrary $S^{(n)}$ of ghost number 0 and for $S_i'^{(n)}$ of ghost number 0 satisfying $\Delta S_i' = 0$ (Asakawa and Kishimoto, hep-th/9909139)

The ambiguity due to S is of a gauge type, the one due to S' is of a covariant type.

Ambiguities in the finite version

The Seiberg-Witten equations

$$\delta\Psi = i\Lambda \star \Psi, \quad \delta\Lambda = i\Lambda \star \Lambda \quad \delta A_i = \partial_i \Lambda - i[A_i \stackrel{\star}{,} \Lambda]$$

are invariant under the noncommutative finite gauge transformations (Stora)

$$\begin{aligned} \Lambda &\to G^{-1}\Lambda G + i\, G^{-1}\delta_v G \\ A_i &\to G^{-1} A_i\, G + i\, G^{-1}\partial_i\, G \\ \Psi &\to G^{-1}\,\Psi \end{aligned}$$

where all products are star products, G is an arbitrary element of ghost number 0.

The gauge ambiguities at the infinitesimal level can be recovered by choosing

$$G = 1 - iS^{(n)}$$

To first order

$$S^{(1)} = -i\theta^{ij}[a_i, a_j]$$

Only for abelian gauge theory there can be an ambiguity of covariant type also for Λ, which however contains v.

The homotopy operator

The consistency condition for the SW map suggests an analogy with the cohomology of chiral anomalies (Zumino, Les Houches lecture).

It is not possible to invert Δ, because it is nilpotent, but it is possible to construct the homotopy operator K satisfying

$$\Delta K + K\Delta = 1$$

Then

$$\Delta KM + K\Delta M = \Delta KM = M$$

and therefore $\Lambda = KM$ is a solution.

Only b_i and its derivatives enter in the equations, and never v itself. K is defined only on b_i.

Construction of K proceeds in two steps.
Basic variables: $a_i,\ b_i$
First, define infinitesimal version L
Action of L:

$$La_i = 0, \quad Lb_i = a_i, \quad [L, D_i] = 0$$
$$L(f_1 f_2) = (Lf_1)f_2 + (-1)^{deg(f_1)} f_1 (Lf_2)$$

It satisfies $L^2 = 0$.
L is odd.
Introduce $d =$ total order(monomial in a, b)
Then the homotopy operator K is defined:

$$K = D^{-1} L$$

with D^{-1} linear operator, which on monomials multiplies by $\frac{1}{d}$
It satisfies $K^2 = 0$.
K is odd and has ghost number -1.
Example:

$$\begin{aligned}\Lambda^{(1)} &= K(-\frac{1}{2}\theta^{ij} b_i b_j) = -\frac{1}{2}\theta^{ij} D^{-1} L(b_i b_j) \\ &= -\frac{1}{2}\theta^{ij} D^{-1}(a_i b_j - b_i a_j) \\ &= \frac{1}{4}\theta^{ij} \left\{ b_i, a_j \right\}\end{aligned}$$

The constraints

The variables a_i and b_i are not free, because from $b_i \equiv \partial_i v$ it follows

$$\partial_i b_j - \partial_j b_i = 0$$

which is equivalent to

$$\Delta F_{ij} = D_i b_j - D_j b_i + i[b_i, a_j] + i[a_i, b_j] = 0$$

Analogously, the covariant derivatives have to satisfy the constraint

$$\left[F_{ij}, \cdot\right] - i[D_i, D_j](\cdot) = 0$$

Solution: Symmetrization procedure
Separate the symmetric part of $D^k a$ or $D^k b$ and substitute the constraints recursively for the antisymmetric pieces. For example:

$$D_i a_j \rightarrow \frac{1}{2}(D_i a_j + D_j a_i + F_{ij} - i[a_i, a_j])$$

Then treat F and its derivatives as scalars.
There are no independent constraints of higher order.

Solutions to second order

By applying the homotopy operator to the symmetrized $M^{(2)}$:

$$
\begin{aligned}
\Lambda^{(2)} \;=\; & -\frac{1}{2}\theta^{ij}\{a_i, \frac{1}{3}D_j\Lambda^{(1)} + \frac{i}{4}[a_j, \Lambda^{(1)}]\} \\
& +\theta^{ij}\theta^{kl}(- \frac{i}{16}[D_i a_k, D_j b_l] \\
& +[[a_i, a_k], \frac{1}{24}D_j b_l + \frac{i}{32}[a_j, b_l]] \\
& +\frac{1}{24}[D_i a_k, [a_j, b_l]] \\
& +\frac{1}{8}(a_i(\frac{1}{3}D_j a_k - \frac{1}{3}D_k a_j + \frac{i}{2}[a_j, a_k])b_l \\
& -b_i(\frac{1}{3}D_j a_k - \frac{1}{3}D_k a_j + \frac{i}{2}[a_j, a_k])a_l \\
& +\{\frac{1}{6}(D_i a_k - D_k a_i) + \frac{i}{4}[a_i, a_k], \{a_l, b_j\}\})) \;.
\end{aligned}
$$

A known solution (Munich group) is

$$
\begin{aligned}
\tilde{\Lambda}^{(2)} \;=\; & \frac{1}{32}\theta^{ij}\theta^{kl}(- \{b_i, \{a_k, i[a_j, a_l] + 4\partial_l a_j\}\} \\
& -i\{a_j, \{a_l, [b_i, a_k]\}\} + 2i[[a_j, a_l], [b_i, a_k]] \\
& +2[[b_i, a_k] + i\partial_i b_k, \partial_j a_l])
\end{aligned}
$$

Comparison between the solutions

As expected, the two solutions $\Lambda^{(2)}$ and $\tilde{\Lambda}^{(2)}$ differ by an ambiguity $\Delta S^{(2)}$ with

$$\begin{aligned} S^{(2)} &= K(\Lambda^{(2)} - \tilde{\Lambda}^{(2)}) \\ &= \theta^{ij}\theta^{kl}[(\frac{1}{24}([a_j,[D_i a_k, a_l]] \\ &\quad +2(D_i a_k a_j a_l + a_l a_j D_i a_k) \\ &\quad +\frac{1}{16}[a_i a_k, \Delta F_{jl}]] \ . \end{aligned}$$

The same technique can be applied to compute the gauge potential $A_i^{(2)}$ and to higher orders in θ^{ij}. It can be done e.g. by computer.

Seiberg-Witten differential equation

Let $\theta \to t\theta$.

The star product depends on an evolution parameter t.

Define new operators at "time" t:

$$\Delta_t = \begin{cases} \delta_v - i\{\Lambda \overset{\star}{,} \cdot\} & \text{on odd quantities} \\ \delta_v - i[\Lambda \overset{\star}{,} \cdot] & \text{on even quantities} \end{cases}$$

Covariant derivative $D_{i,t}$

$$D_{i,t} = \partial_i - i[A_i \overset{\star}{,} \cdot]$$

They have the properties

$$\Delta_t A_i = \partial_i \Lambda, \quad \Delta_t^2 = 0, \quad [\Delta_t, D_{i,t}] = 0$$

$$\Delta_t(f_1 f_2) = (\Delta_t f_1) f_2 + (-1)^{deg(f_1)} f_1 (\Delta_t f_2)$$

Differentiate the Seiberg-Witten equations

$$\begin{aligned} \Delta_t \dot{\Lambda} &= -\theta^{kl} \partial_k \Lambda \star \partial_l \Lambda \\ \Delta_t \dot{A}_i &= \Delta_i \dot{\Lambda} + \frac{1}{2} \theta^{kl} \{\partial_k A_i \overset{\star}{,} \partial_l \Lambda\} \end{aligned}$$

where $\dot{f} = \frac{df}{dt}$

Then a solution are the evolution equations

$$\dot{\Lambda} = \frac{1}{4}\theta^{ij}\left\{\partial_i\Lambda, A_j\right\}$$
$$\dot{A}_i = -\frac{1}{4}\theta^{kl}\left\{A_k, \partial_l A_i + F_{li}\right\}$$

By differentiating $\dot{\Lambda}$

$$\ddot{\Lambda} = \frac{1}{16}\theta^{ij}\theta^{kl}\Big(\left\{\left\{\partial_i\partial_k\Lambda \overset{\star}{,} A_j\right\} + \left\{\partial_i\Lambda \overset{\star}{,} \partial_k A_j\right\} A_l\right\}$$
$$-\left\{\partial_i\Lambda \overset{\star}{,} \left\{A_k \overset{\star}{,} \partial_l A_j + F_{lj}\right\}\right\}$$
$$+2i\left[\partial_i\partial_k\Lambda \overset{\star}{,} \partial_j A_l\right] \Big)$$

We can compute $\frac{d^n\Lambda}{dt^n}$. We can obtain solutions $\Lambda^{(n)}$ as

$$\Lambda^{(n)} = \frac{1}{n!}\frac{d^n\Lambda}{dt^n}$$

The solution to second order obtained from $\ddot{\Lambda}$ again differs from $\Lambda^{(2)}$ by an ambiguity.

With this method the homotopy operator has to be applied at most at first order: no problems with constraints.

Conclusions and outlook

- With a cohomological approach the solutions to the SW equations can be computed for each gauge group and to each order in θ.

- By using ghosts a connection with the Batalin-Vilkoviskij formalism can be made. The SW map could be formulated in terms of a master equation (see Barnich, Grigoriev, Henneaux, hep-th/0106188)

- This type of cohomology is related to an algebroid structure, because it comes from the action of Lie algebra on the fields. Such structure could be used to investigate its properties.

- Through the use of cohomology the renormalization properties of a noncommutative gauge theory could be studied. Expanding in θ is a way to get around the infrared-ultraviolet mixing occurring in noncommutative field theories (see Grosse with Vienna group, hep-th/0104097)

- It is possible to deform the BRST operator δ itself rather than the gauge parameter and the gauge potential. It is an equivalent approach (Weinstein).

- The Weyl-Moyal product appears in string field theory, because it is related to the Witten star product. The SW map may prove relevant in this context.