M-theory on Manifolds with Exceptional Holonomy

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Abstract: In this lecture I review recent progress on construction of manifolds with exceptional holonomy and their application in string theory and M-theory.

1 Introduction

Recently, M-theory compactifications on manifolds of exceptional holonomy have attracted considerable attention. These models allow one to geometrically engineer various minimally supersymmetric gauge theories, which typically have a rich dynamical structure. A particularly interesting aspect of such models is the behaviour near a classical singularity, where one might expect extra massless degrees of freedom, enhancement of gauge symmetry, or a phase transition to a different theory.

In this lecture I will try to explain two interesting problems in this subject — namely, construction of manifolds with exceptional holonomy and the analysis of the physics associated with singularities — and present general methods for their solution. I will also try to make this lecture self-contained and pedagogical, so that no special background is needed. In particular, below we start with an introduction to special holonomy, then explain its relation to minimal supersymmetry and proceed to the main questions.

1.1 Special Holonomy and Supersymmetry

Consider an oriented manifold X of real dimension n and a vector $\vec{v}$ at some point on this manifold. One can explore the geometry of X by doing a parallel transport of $\vec{v}$ along a closed contractible path in X, see Figure 1. Under such an operation the vector $\vec{v}$ may not come back to itself. In fact, generically it will transform into a different vector that depends on the geometry of X, on the path, and on the connection which was used to transport $\vec{v}$. For a Riemannian manifold X, the natural connection is the Levi-Cevita connection. Furthermore, Riemannian geometry also tells us that the length of the vector covariantly transported along a closed path should be the same as the length of the original vector. But the orientation may be different, and this is precisely what we are going to discuss.

The relative orientation of the vector after parallel transport with respect to the orientation of the original vector $\vec{v}$ is described by *holonomy*. On a n-dimensional manifold holonomy is conveniently characterised by an element of the special orthogonal group,

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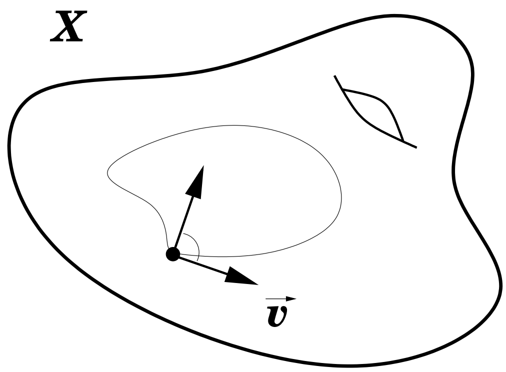

Figure 1: Parallel transport of a vector $\vec{v}$ along a closed path on the manifold X.

$SO(n)$. It is not hard to see that the set of all holonomies themselves form a group, called the *holonomy group*, where the group structure is induced by the composition of paths and an inverse corresponds to a path traversed in the opposite direction. From the way we introduced the holonomy group, $Hol(X)$, it seems to depend on the choice of the point where we start and finish parallel transport. However, for a generic choice of such point the holonomy group does not depend on it, and therefore $Hol(X)$ becomes a true geometric characteristic of the manifold X. By definition, we have

$$Hol(X) \subseteq SO(n) \tag{1}$$

where the equality holds for a generic Riemannian manifold X.

In some special instances, however, one finds that $Hol(X)$ is a proper subgroup of $SO(n)$. In such cases, we say that X is a special holonomy manifold or a manifold with restricted holonomy. These manifolds are in some sense distinguished, for they exhibit nice geometric properties. These properties are typically associated with the existence of non-degenerate (in some suitable sense) p-forms which are covariantly constant. Such p-forms also serve as calibrations, and are related to the subject of minimal varieties.

The possible choices for $Hol(X) \subset SO(n)$ are limited, and were completelty classified by M. Berger in 1955 [1]. Specifically, for X simply-connected and neither locally a product nor symmetric, the only possibilities for $Hol(X)$, other than the generic case of $SO(n)$, are $U\left(\frac{n}{2}\right)$, $SU\left(\frac{n}{2}\right)$, $Sp\left(\frac{n}{4}\right) \times Sp(1)$, $Sp\left(\frac{n}{4}\right)$, $G2$ [1], $Spin(7)$ or $Spin(9)$, see Table 1. The first four of these correspond, respectively, to a Kähler, Calabi-Yau, Quaternionic Kähler or hyper-Kähler manifold. The last three possibilities are the so-called exceptional cases, which occur only in dimensions 7, 8 and 16, respectively. The case of 16-manifold with $Spin(9)$ holonomy is in some sense trivial since the Riemannian metric on any such manifold is always symmetric [2].

Roughly speaking, one can think of the holonomy group as a geometric characteristic of the manifold that tells us how much symmetry this manifold has. Namely, the smaller the holonomy group, the larger the symmetry of the manifold X. Conversely, for manifolds with larger holonomy groups the geometry is less restricted.

This philosophy becomes especially helpful in the physical context of superstring/M-theory compactifications on X. There, the holonomy of X becomes related to the degree of supersymmetry preserved in compactification: the manifolds with larger holonomy group typically preserve smaller fraction of the supersymmetry. This provides a nice link between

[1] The fourteen-dimensional simple Lie group $G2 \subset Spin(7)$ is precisely the automorphism group of the octonions, $\mathbb{O}$.

Metric	Holonomy	Dimension
Kähler	$U\left(\frac{n}{2}\right)$	$n =$ even
Calabi-Yau	$SU\left(\frac{n}{2}\right)$	$n =$ even
HyperKähler	$Sp\left(\frac{n}{4}\right)$	$n =$ multiple of 4
Quaternionic	$Sp\left(\frac{n}{4}\right) Sp(1)$	$n =$ multiple of 4
Exceptional	G_2	7
Exceptional	$Spin(7)$	8
Exceptional	$Spin(9)$	16

Table 1: Berger's list of holonomy groups.

the 'geometric symmetry' (holonomy) and the 'physical symmetry' (supersymmetry). In Table 2 we illustrate this general pattern with a few important examples, which will be used later.

The first example in Table 2 is a torus, T^n, which we view as a quotient of n-dimensional real vector space, $\mathbb{R}^n$, by a lattice. In this example, it is easy to deduce directly from our definition that $X = T^n$ has trivial holonomy group, inhereted from the trivial holonomy of $\mathbb{R}^n$. Indeed, no matter which path we choose on T^n, a parallel transport of a vector $\vec{v}$ along this path always brings it back to itself. Hence, this example is the most symmetric one, in the sense of the previous paragraph, $Hol(X) = \mathbf{1}$. Correspondingly, in string theory toroidal compactifications preserve all of the original supersymmetries.

Our next example is $Hol(X) = SU(3)$ which corresponds to Calabi-Yau manifolds of complex dimension 3 (real dimension 6). These manifolds exhibit a number of remarkable properties, such as mirror symmetry, and are reasonably well studied both in the mathematical and in the physical literature. We just mention here that compactification on Calabi-Yau manifolds preserves 1/4 of the original supersymmetry. In particular, compactification of heterotic string theory on $X = CY_3$ yields $\mathcal{N} = 1$ effective field theory in $3+1$ dimensions.

The last two examples in Table 2 are $G2$ and $Spin(7)$ manifolds; that is, manifolds with holonomy group $G2$ and $Spin(7)$, respectively. They nicely fit into the general pattern, so that as we read Table 2 from left to right the holonomy increases, whereas the fraction of unbroken supersymmetry decreases. Specifically, compactification of M-theory on a manifold with G_2 holonomy leads to $\mathcal{N} = 1$ four-dimensional theory. This is similar to the compactification of heterotic string theory on Calabi-Yau three-folds. However, an advantage of M-theory on G_2 manifolds is that it is completely 'geometric'. Compactification on $Spin(7)$ manifolds breaks supersymmetry even further, to the amount which is too small to be realised in four-dimensional field theory.

Summarising, in Table 2 we listed some examples of special holonomy manifolds that will be discussed below. All of these manifolds preserve a certain fraction of supersymmetry, which depends on the holonomy group. Moreover, all of these manifolds are Ricci-flat,

$$R_{ij} = 0.$$

This useful property guarantees that all backgrounds of the form

$$\mathbb{R}^{11-n} \times X$$

automatically solve eleven-dimensional Einstein equations with vanishing source terms for matter fields.

Of particular interest are M-theory compactifications on manifolds with exceptional

Manifold X	T^n		CY_3		X_{G_2}		$X_{Spin(7)}$
$\dim_{\mathbb{R}}(X)$	n		6		7		8
$Hol(X)$	$\mathbf{1}$	$\subset$	$SU(3)$	$\subset$	G_2	$\subset$	$Spin(7)$
SUSY	1	$>$	1/4	$>$	1/8	$>$	1/16

Table 2: Relation between holonomy and supersymmetry for certain manifolds.

holonomy,

$$\begin{array}{cc} \text{M}-\text{theory on} & \text{M}-\text{theory on} \\ G_2 \text{ manifold} & Spin(7) \text{ manifold} \\ & \\ \Downarrow & \Downarrow \\ & \\ D=3+1 \ \ \text{QFT} & D=2+1 \ \ \text{QFT} \end{array} \tag{2}$$

since they lead to effective theories with minimal supersymmetry in four and three dimensions, respectively. In such theories one can find many interesting phenomena, *e.g.* confinement, various dualities, rich phase structure, non-perturbative effects, *etc.* Moreover, minimal supersymmetry in three and four dimensions (partly) helps with the following important problems:

- The Hierarchy Problem
- The Cosmological Constant Problem
- The Dark Matter Problem

All this makes minimal supersymmetry very attractive and, in particular, motivates the study of M-theory on manifolds with exceptional holonomy. In this context, spectrum of elementary particles in the effective low-energy theory and their interactions are encoded in the geometry of space X. Therefore, understanding the latter may help us to learn more about dynamics of minimally supersymmetric field theories, or even about M-theory itself!

1.2 Why Exceptional Holonomy is Hard

Once we have introduced manifolds with restricted (or special) holonomy, let us try to explain why until recently so little was known about the exceptional cases, G_2 and $Spin(7)$. Indeed, on the physics side, these manifolds are very natural candidates for constructing minimally supersymmetric field theories from string/M-theory compactifications. Therefore, one might expect exceptional holonomy manifolds to be at least as popular and attractive as, say, Calabi-Yau manifolds. However, there are several reasons why exceptional holonomy appeared to be a difficult subject; here we will stress two of them:

- **Existence**

- **Singularities**

Let us now explain each of these problems in turn. The first problem refers to the existence of exceptional holonomy metric on a given manifold X. Namely, it would be useful to have a general theorem which, under some favorable conditions, would guarantee the existence of such a metric. Indeed, the original Berger's classification, described earlier in this section, only tells us which holonomy groups can occur, but says nothing about examples of such manifolds or conditions under which they exist. To illustrate this further, let us recall that when we deal with Calabi-Yau manifolds we use such a theorem all the time — it is a theorem due to Yau (originally, Calabi's conjecture) which guarantees the existence of a Ricci-flat metric on a compact, complex, Kähler manifold X with $c_1(X) = 0$ [3]. Unfortunately, no analogue of this theorem is known in the case of G_2 and $Spin(7)$ holonomy (the local existence of such manifolds was first established in 1985 by Bryant [4]). Therefore, until such a general theorem is found we are limited to a case-by-case analysis of the specific examples. We will return to this problem in the next section.

The second reason why exceptional holonomy manifolds are hard is associated with singularities of these manifolds. As will be explained in more detail in section 2, interesting physics occurs at the singularities. Moreover, the most interesting physics is associated with the types of singularities of maximal codimension, which exploit the geometry of the special holonomy manifold to the fullest. Indeed, singularities with smaller codimension can typically arise in higher dimensional compactifications and, therefore, do not expose peculiar aspects of exceptional holonomy manifolds related to the minimal amount of supersymmetry. Until recently, little was known about these types of degenerations of manifolds with G_2 and $Spin(7)$ holonomy. Moreover, even for known examples of isolated singularities, the dynamics of M-theory in these backgrounds was unclear. Finally, it is important to stress that mathematical understanding of exceptional holonomy manifolds would be incomplete too without proper account of singular limits.

The rest of this lecture consists of two parts which deal with each of these problems in turn.

2 Construction of Manifolds With Exceptional Holonomy

In this section we review various methods of constructing compact and non-compact manifolds with G_2 and $Spin(7)$ holonomy. In the absence of general existence theorems, akin to the Yau's theorem [3], these methods become especially valuable. It is hard to give full justice to all the existing techniques in one section. So, we will try to explain only a few basic methods, focusing mainly on those which played an important role in recent developments in string theory. We also illustrate these general techniques with several concrete examples that will appear in the discussion of singularities and M-theory dynamics.

2.1 Compact Manifolds

The first examples of metrics with $G2$ and $Spin(7)$ holonomy on compact manifolds were constructed by D. Joyce [5]. The basic idea is to start with toroidal orbifolds of the form

$$T^7/\Gamma \qquad \text{or} \qquad T^8/\Gamma \tag{3}$$

where Γ is a finite group, *e.g.* a product of $\mathbb{Z}_2$ cyclic groups. Notice that T^7 and T^8 themselves can be regarded as G_2 and $Spin(7)$ manifolds, respectively. In fact, they

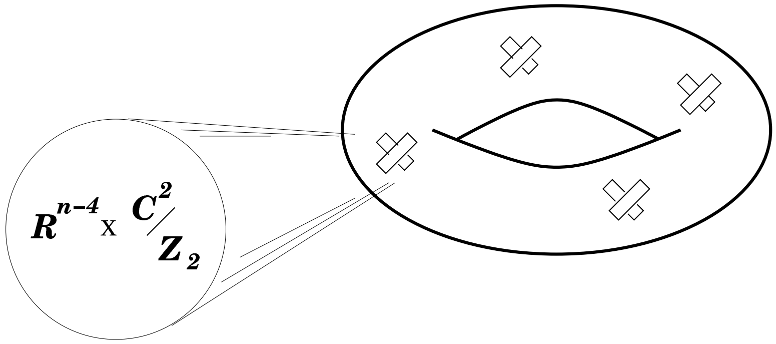

Figure 2: A cartoon representing Joyce orbifold T^n/Γ with $\mathbb{C}^2/\mathbb{Z}_2$ orbifold points.

possess infinitely many G_2 and $Spin(7)$ structures. Therefore, if Γ preserves one of these structures the quotient space automatically will be a manifold with exceptional holonomy.

Example [5]: Consider a torus T^7, parametrised by periodic variables $x_i \sim x_i + 1$, $i = 1, \ldots, 7$. As we pointed out, it admits many G_2 structures. Let us choose one of them:

$$\Phi = e^1\wedge e^2\wedge e^3 + e^1\wedge e^4\wedge e^5 + e^1\wedge e^6\wedge e^7 + e^2\wedge e^4\wedge e^6 - e^2\wedge e^5\wedge e^7 - e^3\wedge e^4\wedge e^7 - e^3\wedge e^5\wedge e^6$$

where $e^j = dx_j$. Furthermore, let us take

$$\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

generated by three involutions

$$\begin{array}{lcl} \alpha & : & (x_1, \ldots, x_7) \mapsto (x_1, x_2, x_3, -x_4, -x_5, -x_6, -x_7) \\ \beta & : & (x_1, \ldots, x_7) \mapsto (x_1, -x_2, -x_3, x_4, x_5, \frac{1}{2} - x_6, -x_7) \\ \gamma & : & (x_1, \ldots, x_7) \mapsto (-x_1, x_2, -x_3, x_4, \frac{1}{2} - x_5, x_6, \frac{1}{2} - x_7) \end{array}$$

It is easy to check that these generators indeed satisfy $\alpha^2 = \beta^2 = \gamma^2 = 1$ and that the group $\Gamma = \langle \alpha, \beta, \gamma \rangle$ preserves the associative three-form Φ given above. It follows that the quotient space $X = T^7/\Gamma$ is a manifold with G_2 holonomy. More precisely, it is an orbifold since the group Γ has fixed points of the form $T^3 \times \mathbb{C}^2/\mathbb{Z}_2$. The existence of orbifold fixed points is a general feature of the Joyce construction.

The quotient space (3) typically has bad (singular) points, as shown in Fig. 2. In order to find a nice manifold X with G_2 or $Spin(7)$ holonomy one has to repair these singularities. In practice, it means removing the local neighbourhood of each singular point and replacing it with a smooth geometry, in a way which does not affect the holonomy group. This may be difficult (or even impossible) for generic orbifold singularities. However, if we have orbifold singularities that can also appear as degenerations of Calabi-Yau manifolds, then things simplify dramatically.

Suppose we have a $\mathbb{Z}_2$ orbifold, as in the previous example:

$$\mathbb{R}^{n-4} \times \mathbb{C}^2/\mathbb{Z}_2$$

where $\mathbb{Z}_2$ acts only on the $\mathbb{C}^2$ factor (by reflecting all the coordiantes). This type of orbifold singularity can be obtained as a singular limit of the A_1 ALE space:

$$\mathbb{R}^{n-4} \times \mathrm{ALE}_{A_1} \quad \to \quad \mathbb{R}^{n-4} \times \mathbb{C}^2/\mathbb{Z}_2$$

Since both ALE space and its singular limit have $SU(2)$ holonomy group they represent local geometry of the $K3$ surface. This is an important point; we used it implicitly to resolve the orbifold singularity with the usual tools from algebraic geometry. Moreover, Joyce proved that resolving orbifold singularities in this way does not change the holonomy group of the quotient space (3). Therefore, by the end of the day, when all singularities are removed, we obtain a smooth, compact manifold X with G_2 or $Spin(7)$ holonomy. In the above example, one finds a smooth manifold X with G_2 holonomy and Betti numbers:

$$b_2(X) = 12, \quad b_3(X) = 43$$

There are many other examples of this construction, which are modelled not only on $K3$ singularities, but also on orbifold singularities of Calabi-Yau three-folds [5]. More examples can be found by replacing tori in (3) by products of the $K3$ surface or Calabi-Yau three-folds with lower-dimensional tori. In such models, finite groups typically act as involutions on $K3$ or Calabi-Yau manifolds, to produce fixed points of a familiar kind. Again, resolving the singularities one finds compact, smooth manifolds with exceptional holonomy.

It may look a little disturbing that in Joyce construction one always finds a compact manifold X with exceptional holonomy near a singular (orbifold) limit. However, from the physics point of view, this is not a problem at all since interesting phenomena usually occur when X develops a singularity. Indeed, compactification on a smooth manifold X whose dimensions are very large (compared to the Planck scale) leads to a very simple effective field theory; it is abelian gauge theory with some number of scalar fields coupled to gravity. To find more interesting physics, such as non-abelian gauge symmetry or chiral matter, one needs singularities.

Moreover, there is a close relation between various types of singularities and the effective physics they produce. A simple, but very important aspect of this relation is that codimension d singularity of X can be associated with physics of $D \geq 11 - d$ dimensional field theory. For example, there is no way one can obtain four-dimensional chiral matter or parity symmetry breaking in $D = 2 + 1$ dimensions from $\mathbb{C}/\mathbb{Z}_2$ singularity in X. Indeed, both of these phenomena are specific to their dimension and can not be lifted to a higher-dimensional theory. Therefore, in order to reproduce them from compactification on X one has to use the geometry of X 'to the fullest' and consider singularities of maximal codimension. This motivates us to study *isolated singular points* in G_2 and $Spin(7)$ manifolds.

Unfortunately, even though Joyce manifolds naturally admit orbifold singularities, none of them contains isolated G_2 or $Spin(7)$ singularities. Indeed, as we explained earlier, it is crucial that orbifold singularities are modelled on Calabi-Yau singularities, so that we can treat them using the familiar methods. Therefore, at best such singularities can give us the same physics as one finds in the corresponding Calabi-Yau manifolds.

Apart from a large class of Joyce manifolds, very few explicit constructions of compact manifolds with exceptional holonomy are known. One nice approach was recently suggested by A. Kovalev [6], where a smooth, compact 7-manifold X with G_2 holonomy is obtained by gluing 'back-to-back' two asymptotically cylindrical Calabi-Yau manifolds W_1 and W_2,

$$X \cong (W_1 \times S^1) \cup (W_2 \times S^1)$$

Although this construction is very elegant, so far it has been limited to very specific types of G_2 manifolds. In particular, it would be interesting to study deformations of these

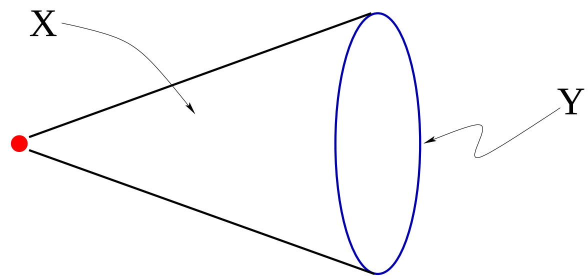

Figure 3: A cone over a compact space Y.

spaces and to see if they can develop isolated singularities interesting in physics. This leaves us with the following

Open Problem: Construct compact G_2 and $Spin(7)$ manifolds with various types of isolated singularities

2.2 Non-compact Manifolds

As we already mentioned earlier, interesting physics occurs at the singular points of the special holonomy manifold X. Depending on the singularity, one may find, for example, extra gauge symmetry, global symmetry, or massless states localized at the singularity. For each type of the singularity, the corresponding physics may be different. However, usually it depends only on the vicinity of the singularity, and not on the rest of the geometry of the space X. Therefore, in order to study the physics associated with a given singularity, one can imagine isolating the local neighbourhood of the singular point and studying it separately. This gives the so-called 'local model' of a singular point. This procedure is similar to considering one gauge factor in the standard model gauge group, rather than studying the whole theory at once. In this sence, non-compact manifolds provide us with basic building blocks of low-energy energy physics that may appear in compactifications on compact manifolds.

Here, we discuss a particular class of isolated singularities, namely *conical singularies*. They correspond to degenerations of the metric on the space X of the form:

$$ds^2(X) = dt^2 + t^2 ds^2(Y), \tag{4}$$

where a compact space Y is the base of the cone; the dimension of Y is one less than the dimension of X. It is clear that X has an isolated singular point at the tip of the cone, except for the special case when Y is a sphere, $\mathbf{S}^{n-1}$, with a round metric.

The conical singularities of the form (4) are among the simplest isolated singularities one could study, see Figure 3. In fact, the first examples of non-compact manifolds with G_2 and $Spin(7)$ holonomy, obtained by Bryant and Salamon [7] and, independently, by Gibbons, Page, and Pope [8], exhibit precisely this type of degeneration. Specifically, the complete metrics constructed in [7, 8] are smooth everywhere, and asymptotically look like (4), for various base manifolds Y. Therefore, they can be considered as resolutions of conical singularities. In Table 3 we list known asymptotically conical (AC) complete metrics with G_2 and $Spin(7)$ holonomy that were originally found in [7, 8] and in more recent literature [9, 10].

Holonomy	Topology of X	Base Y
G_2	$\mathbf{S}^4 \times \mathbb{R}^3$	$\mathbb{C}\mathbf{P}^3$
	$\mathbb{C}\mathbf{P}^2 \times \mathbb{R}^3$	$SU(3)/U(1)^2$
	$\mathbf{S}^3 \times \mathbb{R}^4$	$SU(2) \times SU(2)$
$Spin(7)$	$\mathbf{S}^4 \times \mathbb{R}^4$	$SO(5)/SO(3)$
	$\mathbb{C}\mathbf{P}^2 \times \mathbb{R}^4$	$SU(3)/U(1)$
	$\mathbf{S}^5 \times \mathbb{R}^3$	

Table 3: Asymptotically conical manifolds with G_2 and $Spin(7)$ holonomy.

The method of constructing G_2 and $Spin(7)$ metrics originally used in [7, 8] was essentially based on the direct analysis of the Ricci-flatness equations,

$$R_{ij} = 0, \tag{5}$$

for a particular metric ansatz. We will not go into details of this approach here since it relies on finding the right form of the ansatz and, therefore, is not practical for generalizations. Instead, following [11, 12], we will describe a very powerful approach, recently developed by Hitchin [13], which allows to construct all the G_2 and $Spin(7)$ manifolds listed in Table 3 (and many more !) in a systematic manner. Another advantage of this method is that it leads to first-order differential equations, which are much easier than the second-order Einstein equations (5).

Before we explain the basic idea of Hitchin's construction, notice that for all of the AC manifolds in Table 3 the base manifold Y is a homogeneous quotient space

$$Y = G/K, \tag{6}$$

where G is some group and $K \subset G$ is a subgroup. Therefore, we can think of X as being foliated by *principal orbits* G/K over a positive real line, $\mathbb{R}_+$, as shown on Figure 4. A real variable $t \in \mathbb{R}_+$ in this picture plays the role of the radial coordinate; the best way to see this is from the singular limit, in which the metric on X becomes exactly conical, *cf.* eq. (4).

As we move along $\mathbb{R}_+$, the size and the shape of the principal orbit changes, but only in a way consistent with the symmetries of the coset space G/K. In particular, at some point the principal orbit G/K may collapse into a degenerate orbit,

$$B = G/H \tag{7}$$

where symmetry requires

$$G \supset H \supset K \tag{8}$$

At this point (which we denote $t = t_0$) the "radial evolution" stops, resulting in a non-compact space X with a non-trivial topological cycle B, sometimes called a *bolt*. In other words, the space X is contractible to a compact set B, and from the relation (8) we can easily deduce that the normal space of B inside X is itself a cone on H/K. Therefore, in general, the space X obtained in this way is a singular space, with a conical singularity along the degenerate orbit $B = G/H$. However, if H/K is a round sphere, then the space X is smooth,

$$H/K = \mathbf{S}^k \quad \Longrightarrow \quad X \ \text{smooth}$$

This simply follows from the fact that the normal space of B inside X in such a case is non-singular, $\mathbb{R}^{k+1}$ (= a cone over H/K). It is a good exercise to check that for all

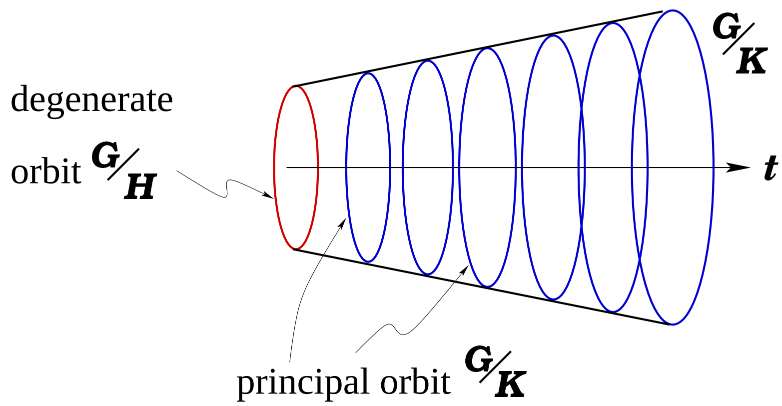

Figure 4: A non-compact space X can be viewed as a foliation by principal orbits $Y = G/K$. The non-trivial cycle in X correspond to the degenerate orbit G/H, where $G \supset H \supset K$.

manifolds listed in Table 3, one indeed has $H/K = \mathbf{S}^k$, for some value of k. To show this, one should first write down the groups G, H, and K, and then find H/K.

The representation of non-compact space X in terms of principal orbits, which are homogeneous coset spaces is very useful. In fact, as we just explained, topology of X simply follows from the group data (8). For example, if $H/K = \mathbf{S}^k$ so that X is smooth, we have

$$X \cong (G/H) \times \mathbb{R}^{k+1} \tag{9}$$

However, this structure can be also used to find a G-nvariant metric on X. In order to do this, all we need to know are the groups G and K.

First, let us sketch the basic idea of Hitchin's construction [13], and then explain the details in some specific examples. For more details and further applications we refer the reader to [11, 12]. We start with a principal orbit $Y = G/K$ which can be, for instance, the base of the conical manifold that we want to construct. Let $\mathcal{P}$ be the space of G-invariant differential forms on Y. It turns out that there exists a symplectic structure on $\mathcal{P}$. This important result allows us to think of the space $\mathcal{P}$ as of the phase space of some dynamical system:

$$\begin{aligned} \mathcal{P} &= \text{ Phase Space} \\ \omega &= \sum dx_i \wedge dp_i \end{aligned} \tag{10}$$

where we parametrised $\mathcal{P}$ by some coordinate variables x_i and the conjugate momentum variables p_i.

Given a principal orbit G/K and a space of G-invariant forms on it, there is a canonical construction of a Hamiltonian $H(x_i, p_i)$ for our dynamical system, such that the Hamiltonian flow equations are equivalent to the special holonomy condition [13]:

$$\left\{ \begin{array}{ccc} \frac{dx_i}{dt} & = & \frac{\partial H}{\partial p_i} \\ \frac{dp_i}{dt} & = & -\frac{\partial H}{\partial x_i} \end{array} \right. \quad \Longleftrightarrow \quad \begin{array}{c} \text{Special Holonomy Metric} \\ \text{on} \quad (t_1, t_2) \times (G/K) \end{array} \tag{11}$$

where the 'time' in the Hamiltonian system is identified with the radial variable t. Thus, solving the Hamiltonian flow equations from $t = t_1$ to $t = t_2$ with a particular boundary condition leads to the special holonomy metric on $(t_1, t_2) \times (G/K)$. Typically, one can extend the boundaries of the interval (t_1, t_2) where the solution is defined to infinity on one side, and to a point $t = t_0$, where the principal orbit degenerates, on the other side. Then, this gives a complete metric with special holonomy on a non-compact manifold X of the form (9). Let us now illustrate these general ideas in more details in a concrete example.

Example: Let us take $G = SU(2)^3$ and $K = SU(2)$. We can form the following natural sequence of subgroups:

$$\begin{array}{ccccc} G & & H & & K \\ \| & & \| & & \| \\ SU(2)^3 & \supset & SU(2)^2 & \supset & SU(2) \end{array} \tag{12}$$

From the general formula (6) it follows that in this example we deal with a space X, whose principal orbits are

$$Y \cong \mathbf{S}^3 \times \mathbf{S}^3$$

Furthermore, $G/H \cong H/K \cong \mathbf{S}^3$ implies that X is a smooth manifold with topology

$$X \cong \mathbf{S}^3 \times \mathbb{R}^4$$

In fact, X is one of the asymptotically conical manifolds listed in Table 3.

In order to find a G_2 metric on this manifold, we need to construct the "phase space", $\mathcal{P}$, that is the space of $SU(2)^3$–invariant 3-forms and 4-forms on $Y = G/K$:

$$\mathcal{P} = \Omega^3_G(G/K) \times \Omega^4_G(G/K)$$

In this example, it turns out that each of the factors is one-dimensional. Therefore, we have only one "coordinate" x and its conjugate "momentum" p. Computing the Hamiltonian $H(x,p)$, we obtain the Hamiltonian flow equations:

$$\left\{ \begin{array}{rcl} \dot{p} & = & x(x-1)^2 \\ \dot{x} & = & p^2 \end{array} \right.$$

These first-order equations can be easily solved, and after a simple change of variables lead to the explicit G_2 metric on the spin bundle over $\mathbf{S}^3$, originally found in [7, 8]:

$$ds^2 = \frac{dr^2}{1 - r_0^2/r^2} + \frac{r^2}{12} \sum_{a=1}^{3} (\sigma_a - \Sigma_a)^2 + \frac{r^2}{36} \left(1 - \frac{r_0^2}{r^2}\right) \sum_{a=1}^{3} (\sigma_a + \Sigma_a)^2 \tag{13}$$

---

3 Topology Change in M-theory

Notice, that all of the non-compact manifolds X with G_2 or $Spin(7)$ holonomy that we discussed in section 2.2 have the form:

$$X \cong B \times (\text{contractible})$$

where B is a non-trivial cycle (a bolt), *e.g.* $B = \mathbf{S}^3$, $\mathbf{S}^4$, $\mathbb{CP}^2$, or something else. Therefore, it is natural to ask: *"What happens if* $\mathrm{Vol}(B) \to 0$ *?"* In this limit the geometry becomes singular. Moreover, if we study string theory or M-theory on X, we might also expect to see some new physics at the singularity, for example,

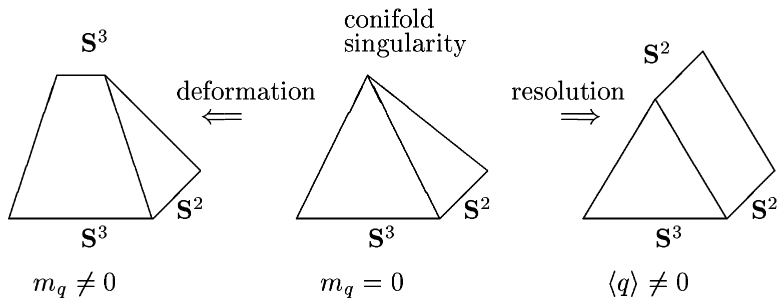

Figure 5: Conifold transition in type IIB string theory.

- New massless objects
- Extra gauge symmetry
- Restoration of continuous/discrete symmetry
- Topology changing transition

  ⋮

In this section we will discuss the latter possibility, namely the situation when one can go to a space with different topology. Although we are mostly interested in exceptional holonomy manifolds, it is instructive to start with topology changing transitions in Calabi-Yau manifolds, where one finds two prototypical examples:

The Flop is a transition between two geometries, where one two-cycle shrinks to a point and a (topologically) different two-cycle grows. This process can be schematically described by the diagram:

$$\mathbf{S}^2_{(1)} \longrightarrow \cdot \longrightarrow \mathbf{S}^2_{(2)}$$

This transition is smooth in string theory [14, 15].

The Conifold transition is another type of topology change, in which a three-cycle shrinks and is replaced by a two-cycle:

$$\mathbf{S}^3 \longrightarrow \cdot \longrightarrow \mathbf{S}^2$$

Ulike the flop, it is a real phase transition in the low-energy dynamics which can be understood as the condensation of massless black holes [16, 17]. Let us briefly recall the main arguments.

As the name indicates, the conifold is a cone over a five dimensional base space which has topology $\mathbf{S}^2 \times \mathbf{S}^3$ (see Figure 5). Two different ways to desingularize this space — called the deformation and the resolution — correspond to replacing the singularity by a finite size $\mathbf{S}^3$ or $\mathbf{S}^2$, respectively. Thus, we have two different spaces, with topology $\mathbf{S}^3 \times \mathbb{R}^3$ and $\mathbf{S}^2 \times \mathbb{R}^4$, which asymptotically look the same.

In type IIB string theory, the two phases of the conifold geometry correspond to different branches in the four-dimensional $\mathcal{N} = 2$ low-energy effective field theory. In the deformed conifold phase, D3-branes wrapped around the 3-sphere give rise to a low-energy field q, with mass determined by the size of the $\mathbf{S}^3$. In the effective four-dimensional supergravity theory these states appear as heavy, point-like, extremal black holes. On the other hand, in the resolved conifold phase the field q acquires an expectation value reflecting the condensation of these black holes. Of course, in order to make the transition from one phase to the other, the field q must become massless somewhere and this happens at the conifold singularity, as illustrated in Figure 5.

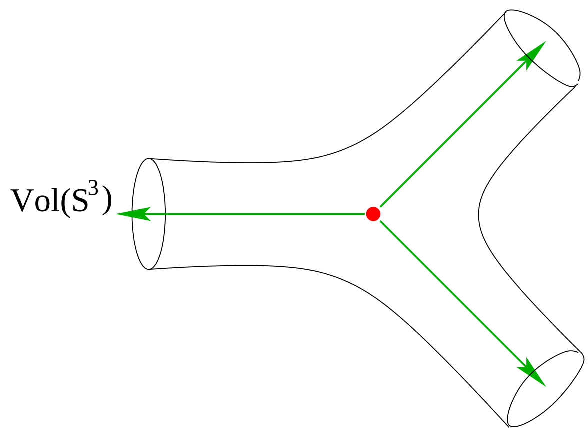

Figure 6: Quantum moduli space of M-theory on G_2 manifold X with topology $\mathbf{S}^3 \times \mathbb{R}^4$. Green lines represent the 'geometric moduli space' parametrised by the volume of the $\mathbf{S}^3$ cycle, which is enlarged to a smooth complex curve by taking into account C-field and quantum effects. The resulting moduli space has three classical limits, which can be connected without passing through the point where geometry becomes singular (represented by red dot in this picture).

Now, let us proceed to topology change in G_2 manifolds. Here, again, one finds two kinds of topology changing transitions, which resemble the flop and the conifold transitions in Calabi-Yau manifolds:

G_2 **Flop** is a transition where 3-cycle collapses and gets replaced by a (topologically) different 3-cycle:

$$\mathbf{S}^3_{(1)} \longrightarrow \cdot \longrightarrow \mathbf{S}^3_{(2)}$$

Note, that this is indeed very similar to the flop transition in Calabi-Yau manifolds, where instead of a 2-cycle we have a 3-cycle shrinking. The physics is also similar, with membranes playing the role of string world-sheet instantons. Remember, that the latter were crucial for flop transition to be smooth in string theory. For a very similar reason, G_2 flop transition is smooth in M-theory. This was first realised by Acharya [18], and by Atiyah, Maldacena, and Vafa [19], for a 7-manifold with topology

$$X \cong \mathbf{S}^3 \times \mathbb{R}^4$$

and studied further by Atiyah and Witten [20]. In particular, they found that M-theory on X has three classical branches, related by triality permutation symmetry, so that the quantum moduli space looks as shown on Figure 6. Once again, important point is that there is no singularity in quantum theory.

Let us proceed to another kind of topology changing transition in manifolds with G_2 holonomy.

A Phase Transition, somewhat similar to a hybrid of the conifold and the G_2 flop transition, can be found in M-theory on a G_2 manifold with topology

$$X \cong \mathbb{CP}^2 \times \mathbb{R}^3$$

A singularity develops when $\mathbb{CP}^2$ cycle shrinks. As in the conifold transition, physics of M-theory on this space also becomes singular at this point. Hence, this is a genuine phase transition [20]. Note, however, that unlike the conifold transition in type IIB string theory, this phase transition is not associated with condensation of any particle-like states

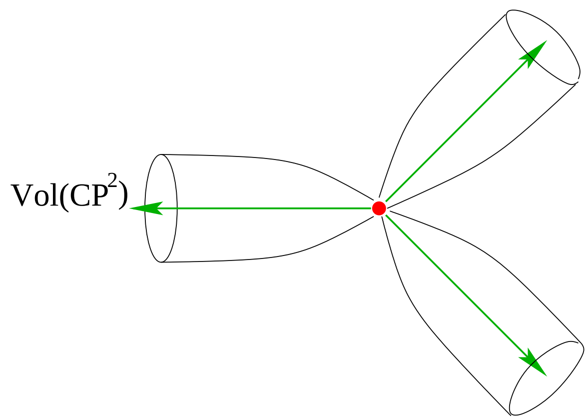

Figure 7: Quantum moduli space of M-theory on G_2 manifold X with topology $\mathbb{C}\mathbf{P}^2 \times \mathbb{R}^3$. Green lines represent the 'geometric moduli space' parametrised by the volume of the $\mathbb{C}\mathbf{P}^2$ cycle, which is enlarged to a *singular* complex curve by taking into account C-field and quantum effects. The resulting moduli space has three classical limits. In order to go from one branch to another one necessarily has to pass through the point where geometry becomes singular (represented by red dot in this picture).

in M-theory[2] on X. Indeed, there are no 4-branes in M-theory, which could result in particle-like objects by wrapping around the collapsing $\mathbb{C}\mathbf{P}^2$ cycle.

Like the G_2 flop transition, this phase transition has three classical branches, which are related by triality symmetry, see Figure 7. Important difference, of course, is that now one can go from one branch to another only through the singular point. In this transition one $\mathbb{C}\mathbf{P}^2$ cycle shrinks and another (topologically different) $\mathbb{C}\mathbf{P}^2$ cycle grows:

$$\mathbb{C}\mathbf{P}^2_{(1)} \longrightarrow \cdot \longrightarrow \mathbb{C}\mathbf{P}^2_{(2)}$$

One way to see that this is indeed the right physics of M-theory on X is to reduce it to type IIA theory with D6-branes in flat space-time [20].

Finally, we come to the last and the hardest case of holonomy groups, namely to $Spin(7)$ holonomy.

$Spin(7)$ **Conifold** is the cone on $SU(3)/U(1)$. It was conjectured in [10] that the effective dynamics of M-theory on the $Spin(7)$ conifold is analogous to that of type IIB string theory on the usual conifold. Namely, $Spin(7)$ cone on $SU(3)/U(1)$ has two different desingularizations, obtained by replacing the conical singularity either wither a 5-sphere or with a $\mathbb{C}\mathbf{P}^2$, see Figure 8. As a result, we obtain two different $Spin(7)$ manifolds, with topology

$$\mathbb{C}\mathbf{P}^2 \times \mathbb{R}^4 \quad \text{and} \quad \mathbf{S}^5 \times \mathbb{R}^3$$

which are connected via topology changing transition

$$\mathbb{C}\mathbf{P}^2 \longrightarrow \cdot \longrightarrow \mathbf{S}^5$$

Moreover, as with the conifold transition [16, 17], the $Spin(7)$ conifold has a nice interpretation in terms of the low-energy effective field theory. Namely, the effective dynamics of M-theory on the cone over $SU(3)/U(1)$ is described by a three-dimensional $\mathcal{N}=1$ abelian Chern-Simons-Higgs theory. The Higgs field q arises upon quantization of the M5-brane wrapped over the $\mathbf{S}^5$. At the conifold point where the five-sphere shrinks, these M5-branes become massless as suggested by the classical geometry. At this point,

[2] However, such interpretation can be given in type IIA string theory [10].

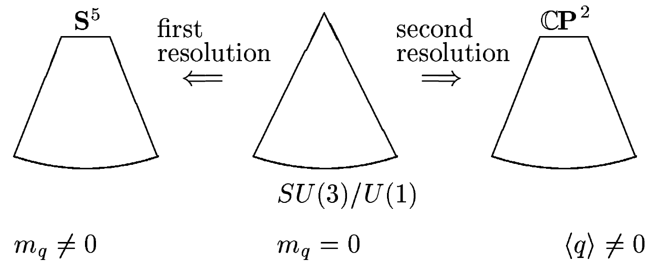

Figure 8: Conifold transition in M-theory on a manifold with $Spin(7)$ holonomy.

the theory may pass through a phase transition into the Higgs phase, associated with the condensation of these five-brane states, see Figure 8.

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