Fluctuations of the Giant Graviton

on

Aspects of Non-commutativity

in ADS/CFT

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t earlier work with

S. Ramgoolam + M. Mihaila-Ru"a
Maldacena's correspondence (AdS/CFT) provides a bridge between large N Yang-Mills type Tc. and supergravity or closed string theory.

Key features of this correspondence are:

\[ \frac{1}{N} \] plays the role of the closed str./gravity coupling constant.

The space-time is given by

\[ \text{AdS} \times S \]

with curvature given by

\[ R \times g^2 N \]
Symmetries are the same on both side (realized differently)

$N(\text{and } SU(n))$ also serves as a cutoff on quantum numbers & emerging space-time

Exclusion P. of Malacena-Strominger

$\frac{1}{12}$ gives the measure of non-commutativity of space-time

Proposal:

$[\text{AdS}]_{\frac{1}{2}} \times [S]_{\frac{1}{2}}$

- Quantum, non-commutative AdS and S

A.J. + S. Ramgoolam

JHEP 9904 (1999)
CFT study of the $\mathcal{S}_N (M)$ orbifold

2d CFT which in the large $N$ limit is related to gravity compactified on $M$.

- The resulting 6d gravity has an $\text{AdS}_3 \times S_3$ background.

- Singular point in mod. space but successful in acc. for entropy of BH.

- Can be used as a lab for systematic study of finite $N$ eft.

hep-th 9902059  Ramgoolan
9907144  Mihailescu
2006...  Mathur’s talk
The ground state of n-twisted sector:

\[ L_0 \mid 0, 6n \rangle = \frac{n^2 - 1}{4m} \mid 0, 6m \rangle \]

\[ J_0 \mid 0, 6m \rangle = 0 \]

Excited states: Chiral primaries

\[ (L_0 - J_0^3) \mid \text{ch. prim} \rangle = 0 \]

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Raise the \( J^3, \bar{J}^3 \) charge of the vacuum

\[ \mid \text{ch}, n \rangle = \frac{n^2 - 1}{4} \psi^+ + \frac{2}{\gamma} \psi^+ - \frac{1}{2} \frac{1}{(\frac{1}{2} + k)} \psi^+ - \frac{1}{2} \frac{1}{(\frac{1}{2} + k)} \]

\[ k = 0 \]

Chiral primaries: States with \( L_0 = J_0 \) are BPS states in gravity

Basic example of ch. prim...
The $S_N$ (permutation) orbifold is a CFT dual of 10D IIB supergravity:

$$AdS_3 \times S_2 \times K_3 \text{ (on } T_4)$$

consists of

$$X^{a,\dot{a}}, \quad \Phi^{a, \dot{a}} \quad a, \dot{a} = 1, 2$$

$$I = 1, \ldots, N \quad S_N \text{ symmetries}$$

$$\Phi(e^{2\pi i \tau}) = q \Phi(\tau) \quad \text{twisted B.C.}$$

$m$-twisted sector

$$X_i^{}(6+2\pi i) = X_{i+2}(6), \quad X_{i+2}(6+2\pi i) = X_i^{} \quad \text{etc.}$$

$n$-bits

Symmetry:

$$SU(2)_L \times SU(2)_R \times SU(N)$$

$$\mathfrak{sl}_2 \times \mathfrak{sl}_2(\mathbb{C})$$

$$J^a, \bar{J}^a \text{ current algebra}$$
A short discussion (work with M. Mihailaescu)

According to the proposal [JR] we are to consider supergravity in a 2-deformed background

\[ \frac{\text{ADS}_3 \times S^3}{\text{S}_3} \]

One can then repeat the calculation of Lee, Minwalla and Seiberg in ADSxS^3 - Mihailaescu ADS_3 x S^3

\[ \phi = \int [\psi (u, t, \varphi)]^N [\psi (m)] \]

The calculation factorizes

\[ \left< \psi^4 \right|_{\text{ADS}} \left< \psi^2, \psi^2, \psi^2 \right|_{S^3} \]

Projected to boundary
Finite $N$ correlations and deformation of space-time

For chiral primary operators, we were able to evaluate finite $N$ operator products (in some folded CFT)

$\Rightarrow$ 3-point correlation functions

For example, pure twist $O^{(0,0)}_{l_3, l_1, l_2}$

\[
\left\langle O^{+}_{l_3}(z) O^{(1)}_{l_2}(\bar{z}) O^{(0)}_{l_1}(\bar{\bar{z}}) \right\rangle \bigg|_{l_1 + l_2 = l_3}
\]

\[
= \frac{(N-l_3)! (N-l_2)!}{(N-l_3)! N! l_1 l_2} \frac{1}{l_3}
\]

- Shows vanishing when $l_3 = l_1 + l_2 > N$
- Conforming with the exclusion P
- Exhibits noncommutativity
$\psi^i_i(n) = D^i_{mn} (\psi, \theta, \psi)$

Star product

$\psi^i_i(n) * \psi^j_j(m) = (i_1 i_2 i_3) \psi^i_j n^k$.

$FC(i, j-i_3, n^n) \psi^i_j$

Fusion Coefficient

$S_3 \ D^i_{mn} = 0$

$\psi \ S_2 \ Y^i_i$

$Tr( Y^i_{i_1 i_2} Y^i_{i_1 i_1} Y^i_{i_2 i_2} )$

$\prod (N-i_1)! (N-i_2)! (N+i_{13}m)!$

$\prod (N-i_3)! (N+i_{13}m)/(N+i_{12}m)$

Same as in the orb. cancel ADS

The structure of 3-point correl.

if consistent with noncomm. (Podles) sphere
McGreevy, Susukiño and Toumbas consider the motion of the "gravitom" on $\text{ADS} \times S$ space. The main effect is coming from a background $p$-form field strength (for $S^p$) by the Kabat-Taylor-Myers mechanism.

The graviton (a collection of D-Branes) grows into a spherically $n-2$ Brane. It is the spherically $n-2$ Brane that Susukiño et al. consider moving on $S^m$. 
We have a spherically symmetric $n$-brane with the radius $R(t)$ and $\phi(t)$ (angle & rotation)

Moving in the induced metric

\[
dS^2 = -dt^2 + \frac{R^2}{R^2 - r^2} dr^2 + (R^2 - r^2) d\Sigma^2 + r^2 d\Omega^2
\]

And the nonzeros (4-form)

\[
F_{\tau \phi 34} = N R r^2
\]

The induced Dirac-Born-Infeld - C-S

Lagrangian reads:

\[
L = -\left( \frac{N}{R^{n-1}} r^{n-2} \right) \sqrt{1 - \left( \frac{R^2}{R^2 - r^2} \dot{r}^2 + (R^2 - r^2) \dot{\phi}^2 \right)} + \frac{N}{(r/R)^{n-1}} \dot{\phi}
\]
Classical solution:

\[ \dot{V} = 0 \quad V = \text{const} \]

\[ \phi \neq 0 \quad \text{nonzero angular momentum} \]

\[ J = \frac{\partial L}{\partial \dot{\phi}} = M(n)(R^{n-1}) \phi + N\left(\frac{n}{R}\right)^{n-1} \]

Solving E-L equation (with \( \dot{V} = 0 \))

\[ n^2 \phi^2 \equiv x \]

\[ x^2 - (n-1)x + (n-2) = 0 \]

\[ \Rightarrow J = N\left(\frac{r}{R}\right)^{n-3} \]

\[ E \cdot R = J \]

The second relation signifies that the solution describes \( B15 \) type state [chiral primaries] in CFT.
The angular momentum formula

\[ J = N \left( \frac{n}{R} \right)^{n-3} \quad 0 \leq R \leq R \]

implies that for:

\[ n = 2 \]

is motion of \( p_n = 0 \) on \( S_2 \) but not on \( S_3 \).

\[ J = N \frac{R}{r} \]

bounded from below.

\[ J \geq N \]

not very good for \( E_8 \).

\[ n = 3 \]

motion of \( p_n = 0 \) on \( S_3 \).

\[ J = N \text{ all } R \]

\[ n > 3 \]

membrane + higher brane.

\[ \text{Exclusion observed} \]

\[ n \leq N \]

(for \( n \) prime.)
• More General Motions:
\[ \dot{\nu} \neq 0 \quad \text{both } \nu(t), \phi(t) \]
Time Dependent.

Useful to go Hamiltonian: \( p_{\nu} = \frac{\partial \mathcal{L}}{\partial \dot{\nu}} \)

\[ J = p_{\phi} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \]

\[ H^2 = m(\nu)^2 + \frac{(J - 2e(\nu))^2}{\bar{R}^2 - r^2} + p_{\nu} \cdot \frac{R^2 - r^2}{R^2} \]

\[ V_{\text{ext}} \]

Scaling:
\[ p_{\nu} = \lambda \bar{p}_{\nu} \]

\[ E^2 = J^2 + \frac{p_{\nu}^2}{2^2 8 \nu} + \frac{(r_j - r_n + 2)^2}{1 - r_n} \]

\[ J = 2 \cdot j \]

\[ \lambda = \frac{\nu}{R r_n} \]

• Implies that for motions such that \( \dot{\nu} \neq 0 \) [i.e. \( E = j \) ]

we have \( E > j \) and no bound on either \( E \) or \( j \)

• The motions are the non-chiral (strong) states
(DUAL) AdS Sphart Graviton

There is more: we have considered the brane expanding in $S^n$ taking just time $t$ from the AdS space-time:

$$\text{AdS} \times S^n$$

$$\downarrow$$

$$dt^2 + R^2 d\bar{X}$$

But one can obviously do the opposite

$$\left.\frac{d^2}{d\tau^2}\right|_{\text{AdS}} + R^2 d\phi$$

only an angle from $S^n$

There is clearly a spherical configuration expanding in AdS space.
$AdS_n$

$\lim_{n \to \infty} = -r^{m-2}\sqrt{1+\frac{r^2}{l^2}\phi^2} + \frac{r^{m-1}}{l}$

\[ J = p_\phi = \frac{\partial L}{\partial \dot{\phi}} \quad \text{Angular momentum in } S_n \]

Two minima

$\nu = 0$ \hspace{1cm} \text{AND} \hspace{1cm} \frac{\nu}{L} = j \frac{1}{m-3}$

*In the AdS case there is NO bound on $j$ since $\nu$ is not bounded

$AdS: \quad (1 + \frac{r^2}{L^2})$ \hspace{1cm} (1 - r^2)

$0 < r < \infty$ \hspace{1cm} of the sphere
• Puzzles (which still remains)

We have three classical configurations with the same BPS energy:

- Giant Graviton Spreading in $S$
- No Dual Giant Graviton Spreading in $A\&S$

• But in the CFT, there is a single state (single ch. primaries or) $E = 3$

Resolution:

Tunneling between the 3 classical vacua lifts the energy of two → single state remains

Still unclear! Julian Lee / 0010191
Fluctuations of the Giant Graviton

we had a spherically symmetric configuration of constant radius $r_0$

Question: Is this conf. stable?

Small fluctuations allow fluctuations in both $\text{AdS}_m \times S^m$ space

Parametrization:

$\text{AdS}_m - (v_0^2 + v_1^2) + v_1^2 + \cdots v_{m-1}^2 = 1$

$v_0 = d\rho \cos \frac{t}{\sqrt{\rho}} + t, \rho$

$v_1 = d\rho \sin \frac{t}{\sqrt{\rho}}$
\[ S_m := L^2 \left( 1 - \frac{r^2}{L^2} \right) d\Phi^2 + \frac{dr^2}{1 - \frac{r^2}{L^2}} + r^2 dS^2_{n-2} \]

\[ r, \phi, X_1, \ldots, X_{n-2} \]

Consider the \( m-2 \) brane, in the physical gauge:
\[ t = t \]
\[ \delta_i = X_i \quad i = 1, \ldots, n-2 \]

The remaining coordinates fluctuate and are general functions \( f(t, \vec{r}) \)

They are:
\[ V(t, \vec{r}) = V_0 + \gamma(t, \vec{r}) \]
\[ \phi(t, \vec{r}) = \omega_0 \tau + \delta \phi \]
\[ V^\mu(t, \vec{r}) \quad \mu = 1, \ldots, m-1 \]
The gravitational background

$$g_{\mu \nu} = \text{AdS}_m \times S_n$$

$$A^{(n-1)} = \frac{\rho^{n-1}}{L}, \quad A^{(n-1)}_{\phi \chi_1 \cdots \chi_{n-2}} = L^{n-1} r^{n-1}$$

For the fluctuating brane: induced Dirac-Bianchi action with a C-S term

$$S = S_{\text{D-B}} + T \int d^4 x \left[ \mathcal{L} \right]$$

Final expression for quadratic fluctuation

$$8S = \int d^4 x d^2 \sigma \left[ \frac{L^2}{2 L^2 - v_0^2} \dot{\sigma}^2 - \frac{L^2}{2(L^2 - v_0^2)} \delta \dot{\sigma} \delta \sigma + \frac{L^2 (L^2 - v_0^2)}{2 v_0^2} \left( \delta \phi \right)^2 - \frac{L^2 (L^2 - v_0^2)}{2 v_0^2} \left( \delta \phi \right) \delta \phi \right.$$ 

$$+ \frac{L^3 (m-3)}{v_0^2} \eta \eta \delta \phi \delta \phi + \frac{L^2}{2 v_0^2} \eta \eta \delta \phi \delta \phi$$

$$+ \frac{L^2}{2} \delta \chi_4 \delta \chi_4 - \frac{2}{2L} \delta \phi \delta \phi \right]$$
Solution:

\[ e^{i \omega t} \gamma (\vec{e}) \alpha \]

\[
\left( \frac{\partial}{\partial \phi} \right) - \text{coupled} \quad \omega_{\pm} = \frac{1}{L^2} \left( \Omega_e + \frac{(n-3)^2}{2} \pm (n-3) \sqrt{\Omega_e^2 + \frac{(n-3)^2}{4}} \right)
\]

\[ \nu_n : \quad \omega^2 = \frac{\Omega_e}{L} + \frac{1}{L^2} \]

Eigenfrequencies. \[ \Omega_e = k (k+1) \]

The excitations of the giant graviton vevs:

\[ E = J + \frac{1}{2} k \omega \]

- Freqs. are positive
- S. S. is stable under small fluct.
**Instanton Solution: Tunneling**

We had an effective Hamiltonian describing radial motions

\[
H = \int \left( \frac{\dot{P}_r^2}{g'(r)} + j^2 + \frac{(jr - r)'^2}{1 - r^2} \right) \sqrt{V(r)} \, dr
\]

- Two minima:
  - \( r_0 = 0 \)
  - \( r_g = \frac{1}{\delta - 1} \)

- Is tunneling between the \( r_0 = 0 \) conf. and the candidate giant gr. \( r_g \) possible?
Yes, there is an instanton solution.

Continue to Euclidean time

\[ S_E = \int d\tau \left[ j^2 + V(r) \right]^{1/4} \left( 1 + g \nu \, \dot{r} \right) \]

"Self-dual" Euclidean equation

\[ \frac{\partial}{\partial \tau} R(\tau) = \pm \frac{1}{p} R(\tau) (j - \nu)^{p-1} \]

Solution: Example $p = 2$ (very simple)

\[ \frac{\nu}{j - \nu} = e^\tau \]

The (Euclidean) action is finite:

\[ S_{\text{inst}} = \left[ j + \frac{1}{2} (1 - j) \log(1-j) - \frac{1}{2} (1+j) \log(1+j) \right] \]
Noncommutativity (as seen by the Giant)

The Giant graviton configuration represents a BPS state where 1/2 of D5 SY is preserved. This reduction can be shown to be equivalent to

\[
Pr = 0
\]

(time independent conf.

Taking this as a reduction of phase space we see the following

\[
\Psi_1 = Pr = 0 \quad \text{primary const.}
\]

\[
\Psi_2 = \frac{dV}{dn} = 0 \quad \text{secondary (}[\Psi_1, H]\])
\]

No further constraints as

\[
[H, \Psi_2] = -\frac{Pr}{H} \frac{V^{(2)}}{Grv} = 0
\]

by \(\Psi_1 = 0\)
Dirac Bracket of $r$ and $\phi$

$$[r, \phi]_D = [r, \phi]_P = 0 \quad \text{(This used to be commutativity)}$$

$$\Rightarrow \quad \frac{\partial^2 V/\partial r \partial \phi}{\partial^2 V/\partial r^2} = \frac{1}{N} \frac{r^{2-p}}{p-1}$$

We see a non-commutativity of sphere

$$[r, \phi] = \frac{1}{N} \frac{r^{2-p}}{p-1}$$

$p = 2$ This is the non-commutative (fuzzy) sphere!
SUMMARY:

We have collected evidence for non-commutativity of AdS_p x Sp space-time(s) in a variety of cases. The non-commutativity parameter is not an external field but the \( \frac{1}{N} \) or \( g_1 \) itself. The cases studied are:

I. 2D non-critical string

\[ \text{AdS}_2 \]

II. Most detailed: \( S_N \) orbifold on \( T^4, K3 \)

\[ \text{AdS}_3 \times [S_3]^{\frac{1}{N}} \text{ noncommutative} \]

III. Semiclassical argument following Susskind, Witten, Perry and Townsend

Giant graviton