

UNC - Duke string theory seminar,
Chapel Hill 08/02/2001

Classical Analysis of a K3 surface related to string theory

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- 1) Introduction
- 2) What is a K3 surface?
(Kummer Construction, Resolution
of an A₁-sing., almost Ricci-flat metric)
- 3) Quantum mechanical approx.
(Large volume limit, Laplace-Operator
on EH space)
- 4) Conclusions and outlook

1) Introduction

- underlying classical field theory of a IIA - superstring theory is formulated as non-linear σ -Modell,

classical action:

$$S = \int_{\Sigma} \text{vol}_{\Sigma} (g_{\mu\nu} - B_{\mu\nu}) \partial X^{\mu} \bar{\partial} X^{\nu}$$

(+ contributions from fermions,
dilaton)

Σ : 2 dim. Riem. mfld., worldsheet

M : target space mfld

$X: \Sigma \rightarrow M$ (immersion)

$$g \in \Gamma(\mathcal{O}^2 T^* M)$$

$$B \in \Gamma(\Lambda^2 T^* M)$$

- Consistency constraints:

- $\dim_{\mathbb{R}} M = 10$

- conformal invariance

- Compactification:

$$M = M_{\text{PR}} \times M_{\text{cpt}}$$

$$\underbrace{(g', B')}_{\text{"space time"}}, \quad \underbrace{(g'', B'')}_{\text{"time space"}}$$

- quantum theory : perturbation expansion with g, B as background fields
- conformal invariance to the first order enforces :

$$\Rightarrow \begin{aligned} g'' &: \text{Ric}(g'') = \text{Tr}(R'') = 0 \\ B'' &: dB'' = 0 \end{aligned}$$
- Def.: A Calabi-Yau mfld. is a cpt. Kähler mfld. with $c_1 = 0$.

Thm. (Yau, 1977)

A cpt. Kähler mfld. is CY
iff there exists a Ricci-flat metric on it.

$$c_1 = \int \frac{1}{2\pi i} \text{Ric} \in H^2(M, \mathbb{C})$$

$$d(\text{Ric}) = 0$$

- Def.: $K3$ is the only nontrivial CY mfld. with $\dim_{\mathbb{C}} = 2$.

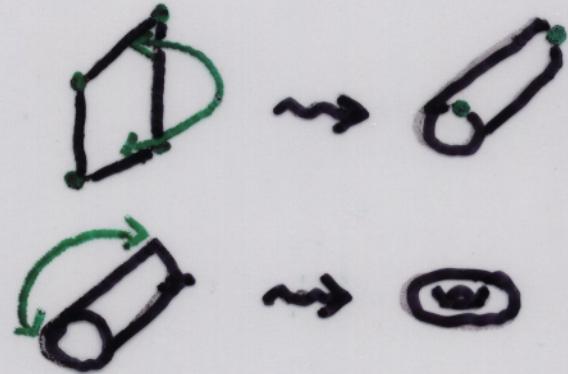
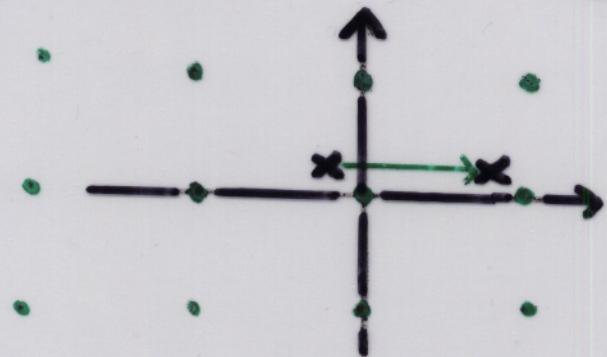
$K3$ is even Hyperkähler
 \Rightarrow Ricci-flat metric is exact vacuum solution of string theory

Problem: none Ricci-flat metric is explicitly known!

2) What is a K3 surface?

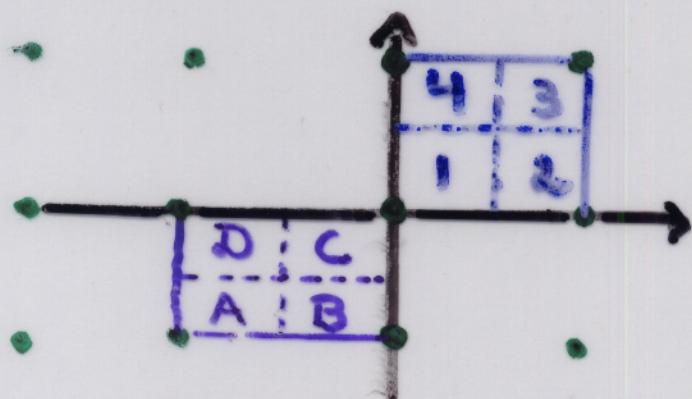
- Kummer Construction (2dim. analogon)

a) Torus : $T^1 = \mathbb{C} / \Gamma$



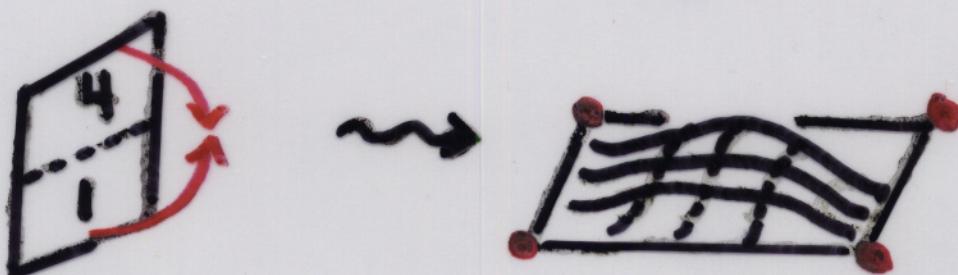
b) \mathbb{Z}_2 -group action

(point reflection at the origin)



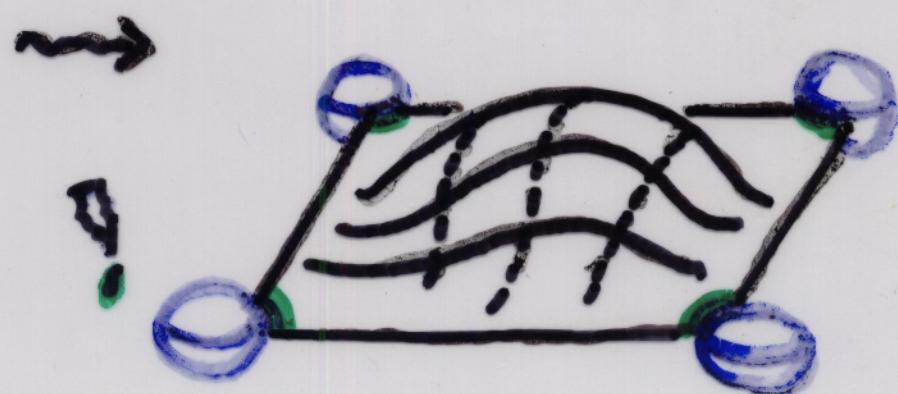
$$\begin{aligned} 2 &\xrightarrow{\mathbb{Z}_2} D \xrightarrow{\Gamma} 4 \\ 3 &\rightarrow A \rightarrow 1 \end{aligned}$$

c) \mathbb{Z}_2 orbifold : T^1 / \mathbb{Z}_2



"Ravioli" with
4 singular points

d) Blown up space



- in the limit of vanishing volume of the balls this becomes a (2 dim. analogon of a) K3 !

2 dim. case	K3
sing. boundary	origin of \mathbb{C}/\mathbb{Z}_2 origin of $\mathbb{C}^2/\mathbb{Z}_2$ S^1/\mathbb{Z}_2 $S^3/\mathbb{Z}_2 \approx \mathbb{RP}^3$

• Resolution of an A_1 -singularity

with the potential W_{A_1} ,

$$W_{A_1}(x, y, z) = xy - (z)^2$$

we can embed $\mathbb{C}^2/\mathbb{Z}_2$ into \mathbb{C}^3

$$\mathbb{C}^2/\mathbb{Z}_2 \approx \{(x, y, z) \in \mathbb{C}^3 \mid W_{A_1}(x, y, z) = 0\}$$

||
A

blow up of \mathbb{C}^3 at the origin

$$\tilde{X} := \{(x, y, z) [s^0 : s^1 : s^2] \in \mathbb{C}^3 \times \mathbb{CP}^2 \mid$$

$$\begin{aligned} x s^1 &= y s^0, & x s^2 &= z s^0, \\ y s^2 &= z s^1 \end{aligned}\}$$

Projections :

$$\pi_1: \tilde{X} \rightarrow \mathbb{C}^3$$

$$\pi_2: \tilde{X} \rightarrow \mathbb{CP}^2$$

equivalent minimal solution:

$$\tilde{A} := \overline{\pi_1^{-1}(A - \{0\})}$$

What Space is \tilde{A} ?

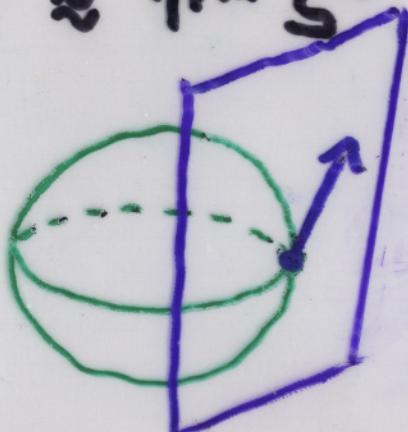
$$\tilde{X} \subset \tilde{A}$$

$$\downarrow \pi_2$$

$$B = \left\{ [s^0 : s^1 : s^2] \in \mathbb{C}\mathbb{P}^2 \mid s^0 s^1 - (s^2)^2 = 0 \right\}$$

$$\dim B = 1$$

$$\tilde{A} \approx T^* S^2$$



$$(u, \xi = y)$$

$$\downarrow$$

$$s^1 \neq 0$$

$$u := \frac{s^2}{s^1}$$

$$(v, \xi = x)$$

$$\downarrow$$

$$s^0 \neq 0$$

$$v := \frac{s^2}{s^0}$$

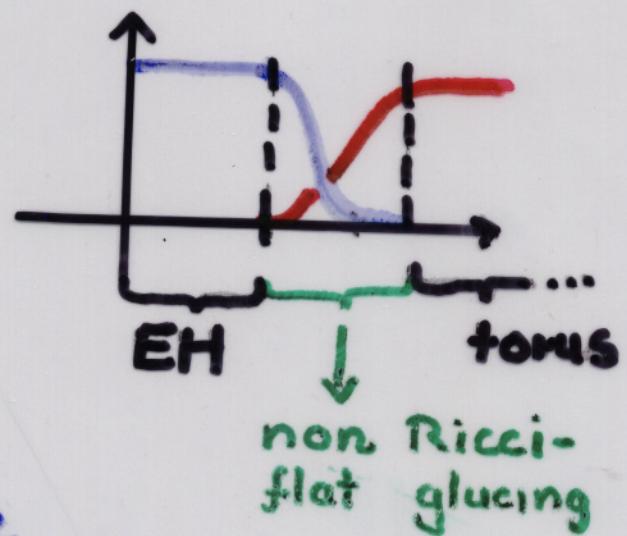
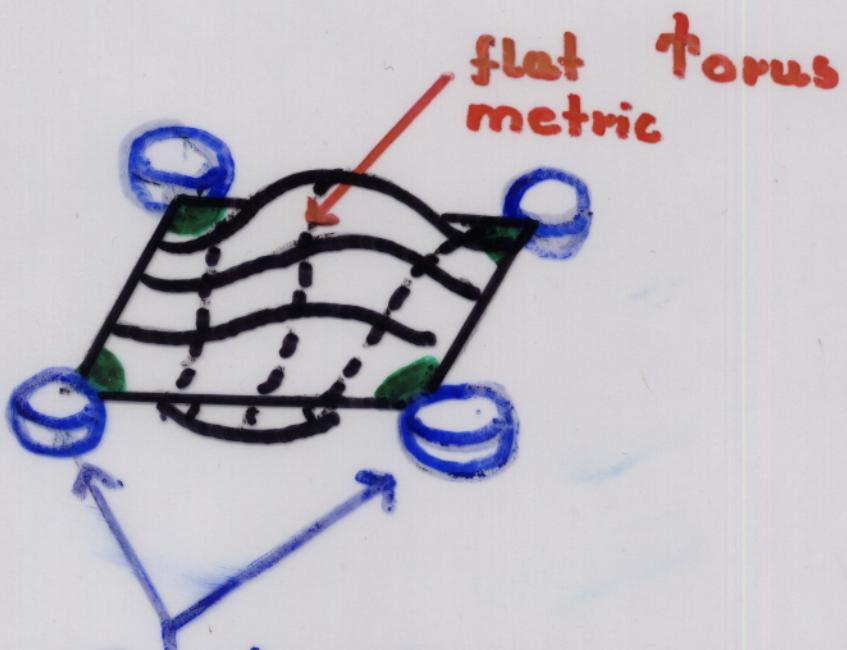
transition fct.

$$(s^1 \neq 0 \neq s^0)$$

- $s^0 s^1 = (s^2)^2 \Rightarrow v = \frac{1}{u}$
- $s^1 x = s^0 y \Rightarrow \xi = u^2 \xi$
- $(v, \xi) = \left(\frac{1}{u}, u^2 \xi \right)$

- singularity has been replaced by $S^2 \approx \mathbb{C}\mathbb{P}^1$

- Page [1981] : Kummer Construction
 \rightsquigarrow Construction of an
almost Ricci - flat metric



- ALE metric on T^*S
 \rightsquigarrow
 - $*R = -R \Rightarrow \text{Ric} = 0$
 - approaches flat metric at infinity
 - fits the boundary
- Taubes [1982], Bozhkov [1988] :
 the limit is a Ricci - flat metric on $K3$!

3) Quantum mechanical approx.

- taking the large volume limit the target space geometry becomes dominant ($QFT \rightsquigarrow QM$)

Hamiltonian: $\hat{H} = -\Delta_{K3}$

- the solutions of

$$\hat{H}\psi = E\psi$$

are the wave functions of strings on $K3$;

E determines the spectrum

- following Page's construction we have to know the solutions of

$$-\Delta_{Eg.-H.}\psi = E\psi \quad (E > 0)$$

and their scattering phase

~ Solutions on $K3$

• Essential part : solutions on EH

$$-\Delta \Psi = E\Psi$$

$SU(2) \times U(1)$

\Rightarrow

$$0 = \left\{ \frac{d}{dz} (z^3 - 1) \frac{d}{dz} + f \right\} \Psi_{rad}$$

$$f = f(\beta = \frac{\alpha^2 E}{4}, j, q)$$

Labels in $SU(2)$ -rep.

Pole structure:

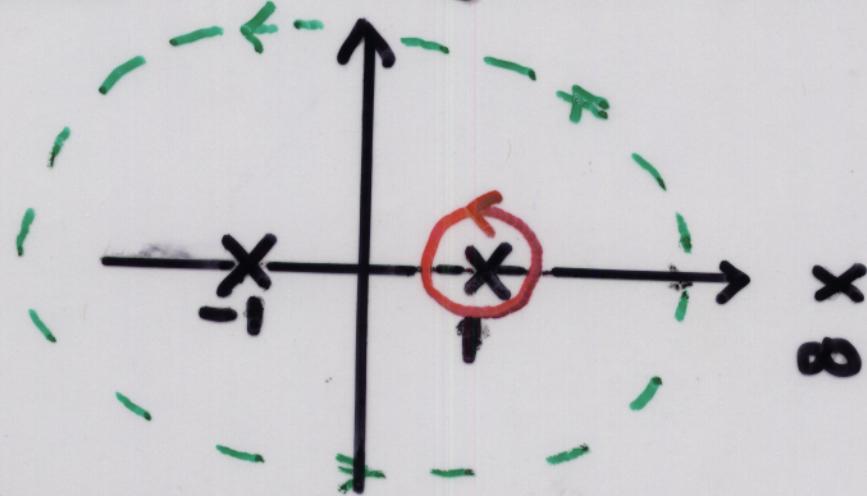
	type	roots	
$z = -1$	regular	$\pm \frac{q}{2}$	unphysical
$z = 1$	regular	$\pm \frac{q}{2}$	Bolt singularity
$z = \infty$	irregular	-	boundary

(generalization of the Bessel and the Hypergeometric ODE)

Aim:

- construct solutions with $\Psi_{rad} \sim z^{q/2}$ for $z \rightarrow 1$
- determine $\Delta_{j,q}(E)$ in $\Psi_{rad} \sim \frac{1}{z^{3/4}} \sin(\sqrt{\beta}z + \Delta_{j,q})$

• Problem: 3 singularities



→ cycles around $z=1$ and $z=\infty$
are not homologous

Alternatives:

a) WKB (Liouville - Green - Approx.)



$$\left\{ \frac{d^2}{dx^2} + (E - V) \right\} \psi = 0$$

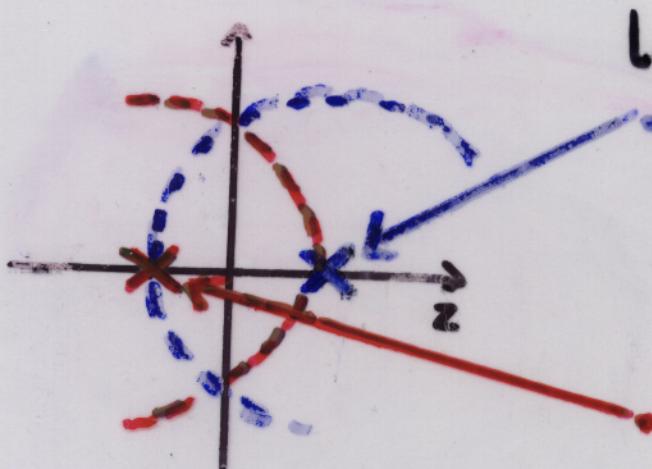
$$x \gg x_0 : \psi \sim e^{\pm i x \sqrt{E-V}}$$

WKB: $x \ll x_0$: solution $e^{i x \sqrt{E-V}}$
becomes recessive $e^{-x \sqrt{E-V}}$

for EH: results of Mignemi [1991]
can be corrected and extended
by results of Olver
only valid for large energies



b) Continued fraction



local expansion:

$$\psi_{\text{reg}} \sim (z-1)^{q/2} \sum_{k=0}^{\infty} a_k (z-1)^k$$

$$\psi_{\text{sing}} \sim \frac{1}{(z-1)^{q/2}} + \dots$$

$$\psi_{\text{reg}} \sim (z+1)^{q/2} \sum_{k=0}^{\infty} b_k (z+1)^k$$

$$\psi_{\text{sing}} \sim \frac{1}{(z+1)^{q/2}} + \dots$$

Problem:

$$\psi_{\text{reg}} \sim \cos(\alpha) \cdot \psi_{\text{reg}} + \sin(\alpha) \cdot \psi_{\text{sing}}$$

\approx $\frac{1}{z^{3/4}} \sin(\sqrt{\beta}z + \delta + \alpha)$

scattering
phase

- for generic values (β, j, q) :
 $\alpha(\beta, j, q) \neq \text{INT}\pi$
 $\Rightarrow \psi_{\text{reg}}$ has radius of convergence $r_c = \infty$

- for special values (β, j, q)
 $\alpha(\beta, j, q) = \text{INT}\pi$
 $\Rightarrow r_c = \infty$

- how do we get these non-generic sets (β, j, q) !

Ψ_{reg} in ODE $\rightarrow a_1 = \delta_0 a_0$

$$\boxed{\forall n : a_{n+1} = \delta_n a_n + \gamma_n a_{n-1}}$$

we are looking for solutions
with

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 0$$

Pincherle's Thm.:

These solutions are given by the zeros of the following meromorphic fct. (in β):

$$F(\beta, j, q) := \delta_0 + \frac{\gamma_1}{\delta_1 + \frac{\gamma_2}{\delta_2 + \frac{\gamma_3}{\ddots}}}$$

for 2 consecutive zeros: β_n, β_{n+1}

$$\Rightarrow \Delta(\beta_{n+1}, j, q) - \Delta(\beta_n, j, q) = \pi$$

$$\Rightarrow \Delta(\beta_n, j, q) = (n-1)\pi + \lim_{N \rightarrow \infty} (\Delta(\beta_N) - N\pi)$$

4) Conclusions

- Construction of a Kummer K3 surface equipped with almost Ricci-flat metric
- in the large volume limit the relevant data of the theory is given by the Laplacian
- we have constructed the eigenfcts and the scattering phase for the EH-space for all energies

Outlook:

- when continuing construction restrictions for value of the B-field? (\rightsquigarrow Aspinwall [1995] Nahm, Wendland [2000])
- Partition fct. of theory via Heat kernel? ($\rightsquigarrow \mathbb{Z}_2$ -orbifold partition fct.)