Warning: I am a mathematician trying to interpret work of several physicists. Beyond my revisionist presentation, there will be some of my joint work with Tom Lada and Ron Pulp of NC State, the noncomm. str. unr which was inspired by work of Behrens, Burgers and van Dam.

In their interpretation of Kontsevich’s proof that any P manifold can be deformation quantized, CBT consider the following \( \sigma \)-model.

To provide a specific example of this correspondence and how it relates to the Batalin-Vilkovisky machinery, we turn to a Poisson sigma model of Cattaneo and Felder [3].

The fields of this Poisson \( \sigma \)-model are ordered pairs \((X, \eta)\) such that \(X\) is a mapping from a 2-dimensional manifold \(\Sigma\) into a Poisson manifold \(M\) and \(\eta\) is a section of the bundle \(\text{Hom}(T\Sigma, X^*\text{T}^*M) \to \Sigma\). These fields are subject to boundary conditions, namely they should satisfy the conditions: \(X(u) = 0\) and \(\eta(u)(v) = 0\) for arbitrary \(u\) in the boundary of \(\Sigma\) and \(v\) tangent to the boundary of \(\Sigma\) at \(u\). Observe that for each \(u \in \Sigma\), we can regard \(\eta(u)\) as a linear mapping from \(T_u\Sigma\) into \(T_{X(u)}\text{M}\). In local coordinates \(\{u^i\}\) on \(\Sigma\) and \(\{x^i\}\) on \(M\), we write \(dX = (dx^i) \frac{\partial}{\partial x^i}\) and \(\eta(\frac{\partial}{\partial x^i}) = \eta_{i\mu}dx^i\). The Poisson structure is given by a Poisson tensor which is a skew-symmetric tensor on \(M\)

\[
\alpha = \alpha^{ij}(\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j})
\]  

(10)

which satisfies a Jacobi condition:

\[
\alpha^{ij}\partial_i\alpha^{jk} + \alpha^{ji}\partial_j\alpha^{ki} + \alpha^{kj}\partial_k\alpha^{ij} = 0,
\]  

(11)
The action $S$ of the model is defined in such local coordinates by

$$S(X, \eta) = \int \left( \eta_i \wedge dX^i \right) + \frac{1}{2}(\alpha^j \circ X)(\eta_i \wedge \eta_j).$$  \hspace{1cm} (12)

The Euler-Lagrange equations are:

$$E_{\eta i} = \frac{\partial L}{\partial \dot{X}^i} - \frac{\partial L}{\partial X^i} = 0$$

The gauge symmetries of the action are parameterized by all sections $\beta$ of the bundle $X^* T^* M \to \Sigma$ which vanish on the boundary of $\Sigma$.

For each such $\beta$, define $\delta_{\beta}$ acting on the fields by

$$(\delta_{\beta} X)^i = (\alpha \circ X)(dx^i, \beta)$$

and similarly for $(\delta_{\beta} \eta)_i$.

\begin{equation}
\delta S = x^i \cdot (X) \beta_j \frac{D}{dx^i} \left[ (\delta_{\beta} \eta)_j - \frac{\partial L}{\partial X^i} + \frac{\partial L}{\partial \dot{X}^i} \right]
\end{equation}

Now Emmy Noether had two variational theorems. The first relates symmetries to conserved quantities as is the better known. The second relates symmetries to relations among the EL equations.

In this example, the relations corresponding to $\delta S$ are:

$$x^i \cdot E_{\dot{X}^i} + \delta \eta_i - \frac{\partial L}{\partial \dot{X}^i} \eta_j; \bar{E}_j = 0$$
5. First steps of the Batalin-Vilkovisky formalism

Rather than review the Batalin-Vilkovisky formalism in general as in [5, 2, 1], we illustrate it by example: the Poisson sigma model we have been considering. Batalin and Vilkovisky first construct a graded commutative algebra over \( \mathbb{R} \) with generators \( X^i_t \) and \( \eta^a \), called 'anti-fields', \( \gamma^i \) called 'ghosts' and \( \gamma^a \), called 'anti-ghosts. (If only the ghosts were used as generators, this would be a BRST algebra.)

By graded commutative, we mean polynomials in the even variables and exterior in the odd.

These generators are bigraded, as indicated in the following table where the form degree is displayed as the top row and the ghost degree as the first column. The graded commutativity is with respect to the sum of the ghost degree and the form degree (which we call the total degree).

The assignments of degree (from left to right) and ghost number (from top to bottom) are given by

\[
\begin{array}{ccc}
0 & 1 & 2 \\
-2 & \gamma^i & \\
-1 & \eta^a & X^i_t \\
0 & \eta_t & \gamma^i \\
1 & \gamma^a & \\
\end{array}
\]

What's going on here? First let's look at the ghosts. If we adjoint just the ghosts, we can define a BRST operator aka the Chevalley-Eilenberg differential:

\[
\begin{align*}
\delta X^i &= \alpha^i(X) \gamma_j, \\
\delta \eta_t &= -d \gamma_t - \partial_i \alpha^{j_k}(X) \eta_j \gamma_k, \\
\delta \gamma^i &= \frac{1}{2} \partial_i \alpha^{j_k}(X) \gamma_j \gamma_k.
\end{align*}
\]

If \( x^{i j}(X) = x^{i k} x^{k j} \), we would be seeing a Lie algebra \( \mathfrak{g} \) with basis \( Z^i \) added to the \( \gamma^i \).
What is a Lie algebra?

There are several current descriptions which Lie might not recognize. Mathematicians prefer a coordinate free version, but physicists seem to like bases. Let \( \mathfrak{g} \) be a basis for \( \mathfrak{g} \).

The algebra structure corresponds to a bracket in terms of structure constants:

\[
\sum \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k = c_{ijk} \mathbf{e}_k
\]

satisfying the Jacobi identity.

Similarly, a representation of \( \mathfrak{g} \) is a vector space \( V \) with basis \( \{ \mathbf{y}_i \} \) and structure constants:

\[
\sum \mathbf{y}_i \mathbf{y}_j \mathbf{y}_k = a_{ijk} \mathbf{y}_k
\]

satisfying an analog of Jacobi.

Define \( \{ \xi_i \} \) to be a dual basis to the \( \mathbf{e}_i \)

The above can be transcribed as:

\[
\sum \xi_i \xi_j = \frac{1}{2} c_{ij} \xi_i \xi_j
\]

\[
\sum \mathbf{y}_i \mathbf{y}_j \mathbf{y}_k = a_{ijk} \mathbf{y}_k
\]

The Jacobi algebra conditions hold if \( \xi^2 = 0 \)

\( \xi \) is called a BRST operator or CE edit.
But in our 5-model, we have structure
fins not constant and so \( S^2 \neq 0 \) occurs
but all is not lost. The great discovery
of 5V was how to work around this by
adding terms of higher order by
introducing anti-fields and antighosts.
The anti-fields correspond to the shell, the
Then we define another differential due to Big
means the anti-fields generate the Koszul-complex with

\[
\begin{align*}
 d_{KT} X_i^+ &= d\eta_i + \frac{1}{2} \partial_i \alpha^{ij}(X) \eta_j \land \eta_i = E_x^i, \\
 d_{KT} \eta^{+i} &= -dX^i - \alpha^{ij}(X) \eta_j = E_{\eta}.
\end{align*}
\] (31)

Because of the Noether identities, the Koszul complex has non-trivial
cohomology in ghost degree \(-1\), namely the classes given by the
formulas for the identities with \( E_x \) and \( E_{\eta} \) replaced by \( X_i^+ \) and \( \eta^{+i} \):

\[
-\alpha^{ij}(X) X_j^+ - \partial_i \alpha^{ij}(X) \eta_j \land \eta^{+k} - d\eta^{+i}.
\] (32)

These classes can be killed by adjoining the anti-ghosts \( \gamma^{+i} \) and defining

\[
 d_{KT} \gamma^{+i} = -\alpha^{ij}(X) X_j^+ - \partial_i \alpha^{ij}(X) \eta_j \land \eta^{+k} - d\eta^{+i}.
\] (33)

Thus the anti-ghosts occur precisely because of the identities identified
by Noether. The second stage is due to Tate.

The pairing between symmetries and identities is now expressed as
the pairing between ghosts and anti-ghosts, which plays a crucial role in
the Batalin-Vilkovisky anti-bracket, but first the anti-fields and anti-
ghosts are themselves subject to symmetries corresponding to \( \delta_\theta \) as follows:

\[
\begin{align*}
 \delta X_i^+ &= \partial_i \alpha^{kj}(X) X_k^+ \gamma_j, \\
 \delta \eta^{+i} &= \partial_i \alpha^{ij}(X) \eta^{+k} \gamma_j, \\
 \delta \gamma^{+i} &= \partial_i \alpha^{ij}(X) \gamma^{+k} \gamma_j.
\end{align*}
\] (34)

6. THE BATALIN-VILKOVISKY ANTI-BRACKET AND TOTAL DIFFERENTIAL

The hoped for total differential \( D \) will be obtained by adding 'terms
of higher order' to \( d_{KT} + \delta \), which does not square to zero. To do this in
general, Batalin and Vilkovisky introduce an 'anti-bracket' ( , ) which
is defined in terms of distributional derivatives of functionals of the
fields and anti-fields.
The pairing defines the antibracket on generators: 
\[ (X^i, X^j_\pm) = \delta^i_\pm \] 
\[ (\eta, \eta^i_\pm) = \delta^i_\pm \] 
\[ (\gamma^i_\pm, \gamma^i_\pm) = \delta^i_\pm \]  

(35)

The BV anti-bracket extends this as a graded biderivation with respect to ghost degree and in this example can be written as \((A, B) = \) 

\[ \sum_{\alpha} \int_\Sigma (-1)^{\phi_\alpha + |d| + |A|} \left( \frac{\partial A}{\partial \phi^\alpha} \wedge \frac{\partial B}{\partial \phi^\alpha} - (-1)^{\deg(d_A + |A| + 1)} \frac{\partial A}{\partial \phi^+} \wedge \frac{\partial B}{\partial \phi^+} \right) \] 

(36)

where \(|C| = gh(C) + deg(C)\) denotes the Grassman parity of \(C\) (\(C\) is either a field or a function of fields). Note that physicists prefer to use both left and right derivatives and hence exhibit a different set of signs.

The antibracket obeys the graded commutativity relation

\[ (A, B) = \frac{1}{(-1)^{gh(A) - 1}(gh(B) - 1)} (B, A) \]

and the Leibnitz rule

\[ (A, BC) = (A, B)C + \frac{1}{(-1)^{gh(A) - 1}gh(B)} B(A, C), \] 

(37)

which emphasizes the resemblance to a Poisson bracket. The only difference from a graded Poisson bracket is that the bracket shifts the degree by 1 and the several identities (skew-commutativity, Jacobi and Leibniz) inherit certain signs. Such an ‘odd’ Poisson bracket is also known as a Gerstenhaber bracket [4].

Now it is possible to express \(d_{KT} + \delta\) in the form \((S^0 + S^1, \quad )\) where

\[ S^0 = (X, \eta) = \int_\Sigma (\eta_i \wedge dX^i) + \frac{1}{2} (\alpha^{ij} \circ X)(\eta_i \wedge \eta_j), \]

our original action, and \(S^1\) is

\[ \int_\Sigma X^i \alpha^{ij}(X) \eta_j - \eta^{+i} \wedge (d\eta_i + \partial_i \alpha^{kj}(X)\eta_k \eta_j) - \frac{1}{2} \gamma^{+i} \partial_i \alpha^{jk}(X) \gamma_j \eta_k \] 

(38)

Corresponding to the fact that \((d_{KT} + \delta)^2 \neq 0\), we have 

\[(S^0 + S^1, S^0 + S^1) \neq 0.\]

The gauge symmetries of the \(\bar{E}_8\) cohomologically incorporate
The additional terms in the differential $D$ we seek will be found by extending $S^0 + S^1$ by terms of higher order to achieve the full BV action $S_{\text{BV}}$.

Batalin and Vilkovisky show that, in much more general situations, one can add terms $S^i$ of ghost degree $i > 1$ to achieve a total $S_{\text{BV}}$ such that

$$(S_{\text{BV}}, S_{\text{BV}}) = 0.$$ 

The reason for this is that the $d_{\text{KT}}$ homology vanishes in appropriate degrees.

In the Cattaneo-Felder model, only one more term is needed:

$$S^2 = \int_S \frac{1}{4} \eta^{ij} \eta^{kl} \partial_i \partial_j \alpha^{kl}(X) \gamma_k \gamma_l.$$ 

Thus the total Batalin-Vilkovisky generator is

$$S_{\text{BV}} = \int_S \eta_i \wedge dX^i + \frac{1}{2} \alpha^{ij}(X) \eta_i \wedge \eta_j$$ 

$$+ X_i^+ \alpha^{ij}(X) \gamma_j - \eta^{ij} \wedge (d \gamma_i + \partial_i \alpha^{kl}(X) \eta_k \gamma_l) - \frac{1}{2} \gamma^i \partial_i \alpha^{jk}(X) \gamma_j \gamma_k$$ 

$$- \frac{1}{4} \eta^{ij} \eta^{kl} \partial_i \partial_j \alpha^{kl}(X) \gamma_k \gamma_l.$$ 

We can then work out the formulas for $D$ applied to the 6 kinds of generators and piece out the formulas for $\delta$, $d_{\text{KT}}$, and more.

But what does more signify and how does this relate to more classic examples of gauge symmetries without field dependence as in [BRvD]?

Here's an alternative that looks better.

Cf. ~
Recall that in total degree $0$ we have $x_i, \eta_i, \gamma_i$ and in total degree $1$ $\xi_i, \nu_i$ and $x_i$.

Captures and Felder thought to take the corresponding sums

$$\bar{x}_i = x_i + \eta_i + \gamma_i$$
$$\bar{x}_i = \xi_i + \nu_i + x_i$$

The total $\delta$ corresponding to $S_{15}$ now looks like

$$\delta \bar{x}_i = D x_i + \alpha(x) \bar{x}_i$$
$$\delta \bar{x}_i = D \bar{x}_i + \frac{1}{2} \alpha(x) \bar{x}_i$$

where $D$ is the de Rham differential on $\Sigma$, but now $\delta^2 = 0$.

And now we see exactly the formalism of BBD for handling field dependent gauge symmetries. They would write

$$\delta \bar{x} = T$$

$$\delta \bar{x} = T(x, \bar{x}) = \sum T_n(x, \ldots, x, \bar{x})$$

and

$$\delta \bar{x}, \delta \bar{x}$$
Their requirements on the relations among $T \in \mathfrak{g}$ are precisely to $\xi^2 = 0$.

Here's a final way to look at what's going on.

There's a notion of graded Lie algebra which means the underlying vector space is a sum of pieces $\mathfrak{g} = \bigoplus \mathfrak{g}_n$ or

if you prefer super Lie algebras have $\mathfrak{g} = \text{odd} \oplus \text{even}$. In either case

$$\Sigma, \mathcal{T} : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$$

satisfies a version of Jacobi, with some signs. This data packs up nicely if we consider $\mathfrak{g}^+ = \text{grade by 1}$ and take the graded symmetric algebra. Then extend the bracket as a coderivation $\Delta$

If $\mathfrak{g}$ has a differential $\delta^2 = 0$ encodes Jacobi

If $\mathfrak{g}$ has a differential of its own

RU(3)
unshuffled of \( v_i, \ldots, v_1 \) as usual in superalgebra.

\[ \{ 1-\gamma, \ldots, \gamma \} = \frac{1}{\gamma} \sum_{0 \leq 2 < 1+u\lambda} [u_1, \ldots, \gamma] \sum_{\gamma < 1+u\lambda} \mathcal{O} \]

\[ \{ 1-\gamma, \ldots, \gamma \} = \frac{1}{\gamma} \sum_{0 \leq 2 < 1+u\lambda} [u_1, \ldots, \gamma] \sum_{\gamma < 1+u\lambda} \mathcal{O} \]

\[ \{ 1-\gamma, \ldots, \gamma \} = \frac{1}{\gamma} \sum_{0 \leq 2 < 1+u\lambda} [u_1, \ldots, \gamma] \sum_{\gamma < 1+u\lambda} \mathcal{O} \]

where \( \mathcal{O} \) is the sign picked up by the elements \( v_i \) passing through the unshuffled of \( v_i, \ldots, v_1 \) during the process of taking through \( v_i \).

This defines \( \gamma \) and satisfies the relations

\[ [u_1, \ldots, \gamma, 1+u\lambda, \ldots, 1\lambda] |_{1+u\lambda} |_{1\lambda} (1-\gamma) = [u_1, \ldots, \gamma, 1+u\lambda, \ldots, 1\lambda] \]

which are homogeneous of degree \( 2n \) and super (or graded) symmetric.

\[ \{ u_1, \ldots, \gamma, 1+u\lambda, \ldots, 1\lambda \} |_{1+u\lambda} |_{1\lambda} (1-\gamma) = [u_1, \ldots, \gamma, 1+u\lambda, \ldots, 1\lambda] \]

This defines \( \Lambda \) and a collection of \( u \)-ary brackets.

\[ \Lambda \]

with a differential \( \partial \), \( \partial = 0 \), of degree 1 and a collection of \( nu \)-ary algebras. The algebra is a complex structure.

Definition 1.3 (Homotopy Lie algebras). A

...
which is graded skew \[ X \cdot Y = -(-1)^{x+y} Y \cdot X \]
where \((-1)^{x+y} \) means \((-1)^{deg X} \cdot (-1)^{deg Y} \)

and satisfies Jacobi signs, easiest to remember is to perform

The CE complex needs only minor modification

Define \( \uparrow g = s_0 \cdot g = [\uparrow g] \) by

\[
(\uparrow g)_{n+1} = \mathcal{E}_{n}
\]

Let \( A^\otimes \) for a graded vector space \( V \)

let \( \Lambda \cdot V = \text{graded algebra generated by} V \)

\[ \approx A^\otimes \otimes S(V^\otimes) \]

Define a coderivation on \( \Lambda \cdot \uparrow g \)

by \( s(\uparrow x) = -\uparrow x \cdot x \)

\[ s(\uparrow x \cdot \uparrow y) = \uparrow [x, y] \]

and extend as a coder

i.e.,

\[
s(\uparrow x_1 \cdot \ldots \cdot \uparrow x_n) = \sum \uparrow x_1 \cdot \ldots \cdot \uparrow x_n
\]

\[ s^2 = 0\]

Expressed that way, one could long ago have asked what if we add terms \( s(\uparrow x_1 \cdot \ldots \cdot \uparrow x_n) = \uparrow [x_1, \ldots, x_n] \)

and still asked for \( s^2 = 0 \)

Now historically that's not what happened but

UNC That's what hiding in BV
Rather the associative analog arose in topology in terms of the chains on a based loop space and then in the definition of rational homotopy types.

Let's cut to the chase and analyze the pieces of such a $D$.

Call the pieces $D_n$ which correspond $\Rightarrow D_1 \Rightarrow 0 \Rightarrow D_2 \Rightarrow 0 \Rightarrow d^2 = 0$

$D^2 = 0 \Rightarrow D_1 \Rightarrow 0 \Rightarrow D_2 \Rightarrow 0 \Rightarrow d^2 = 0$

$\Rightarrow D_1 D_2 \Rightarrow d_2 D_1 = 0$

i.e. $D_2$ is still a chain map.

$D_1$ is a derivation of $E, I$

but something new next.

Instead of Tac $\Rightarrow d_2 D_2 = 0$

we now have $D_2 D_2 = D_1 D_3 + D_3 D_1$

so $\text{Tac}(x, y, z) = d^2(I_{x^1, y^1, z^1} + \int d^2 x, y, z)$

Jacobi form of $\varepsilon$ closed element fails by an "exact term"

$\text{Tacobi}$ holds up to $\varepsilon$.
Notice that the Lax algebra is $L = L_0 \oplus L_1$ where $L_i \subseteq L_0$ and $x^i \in L_i$ and $y^j \in L_0$ but $L_0$ is not a subalgebra.

What is the physical significance of the individual terms?

First write $S_{1 2} = \sum_1^N \delta x^i \pm A(i, x) \mp X_j$

then expand $A(i, x)$ to see $n$-point functions for field interactions. This works wherever BV applies, e.g., to all the BBV examples.

Another very important case is the physically significant example of an Lax structure is in Zweibach's CFT where the fields are functions on the space of closed strings and the convolution product of such fields corresponds to a closed string decomposition.
\[ [\phi, \psi](x_0) = \iiint \phi(x_1) \psi(x_2) \, dx \]
Here the higher order Loo brackets correspond to n-point functions determined by further decompositions e.g.:

\[ \begin{array}{c}
\text{a} \\
\text{b} \\
\text{c}
\end{array} \]

Where only if a or b or c = \pi is the result an iterated \[ \Sigma, I, J \].

Furthermore!
Quite a different place where Loo algebras appear naturally is in field theory as in my work with Mike Schlesinger and in his def q which brings us full circle to the beginning of this talk.