Motivations

"Separation of Variables"
for classical systems
(SoV)

$(M, \omega)$ symplectic manifold

$(P_1, \ldots, P_n, q_1, \ldots, q_n)$

$\omega_{ij} = \sum_{i=1}^{n} dq_i \wedge dp_i$

Hamilton \[ H(q_1, \ldots, q_n, \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_n}) = E \]

Jacobi

It is called separable in $(q, \dot{q})$ if $HJ$ admits complete integral

$S(q_1, \ldots, q_n) = \sum_{i=1}^{n} S_i(q_i, \dot{q}_i, \ldots, \dot{q}_n)$

Complete means a solution

$S = S(q_1, \ldots, q_n)$

di parameters $|\frac{\partial S}{\partial \dot{q}_i}| \neq 0$
Integrable system $$(H_1, \ldots, H_n)$$

$$H_i: M \to \mathbb{R}$$

$$\{H_i, H_j\} = 0$$

$$\partial H_1 \wedge \cdots \wedge \partial H_n \neq 0$$

is separable in coordinates $$(\overline{p}_i, \overline{q}^i)$$ if we have non-trivial relations

$$\overline{H}_i (q_i, p_i; H_1, \ldots, H_n) = 0 \quad i = 1, \ldots, n$$

come in pairs

$$p_i = p_i (q_i; H_1, \ldots, H_n) \quad i = 1, \ldots, n$$

$$s = s (\overline{q}^i; \alpha_1, \ldots, \alpha_n) = \sum_{i=1}^{n} \int \overline{p}_i (q_i; H_1, \ldots, H_n) \, dq_i$$

Note:

1. $$H_i = \alpha_i$$ define Lagrangian foliation \( \Sigma \)

2. $$dS = \Theta / \Sigma$$

Note non-geometric:

3. $$\Phi_i$$'s have the special property to contain a single pair of canonical coordinates at time.
Find intrinsic conditions on \((H_1, \ldots, H_n)\) to ensure separability in a set of coordinates.

**Bihamiltonian Approach**

**Definition**

A \(C^\infty\) manifold \(M\) is bihamiltonian if \(P_1, P_2 \in \Gamma(\Lambda^2 TM)\) Poisson tensors

\[
(P_i(df, dg) = \{f, g\}_i, \quad i = 1, 2)
\]

1. \(P_1 + \lambda P_2\) is Poisson for all \(\lambda \in \mathbb{R}\)
2. \(P_2\) is invertible

\[
\mathcal{N} = P_1 P_2^{-1} \in \Gamma(\text{End} TM)
\]

\[
[N_x, N_y] - \mathcal{N}([N_x, y] + [x, N_y] - N[x, y]) = 0
\]

\(\forall x, y \in \Gamma(TM)\)

**Theorem**

On \(M\) we have coordinates \((x^i, y_i)\)

1. \(P_2 = \omega = \sum_{i=1}^{n} dy_i \wedge dx_i\)
2. \(N = \sum_{i=1}^{n} \lambda_i \left( \frac{\partial}{\partial x_i} \right) + \frac{\partial}{\partial y_i} \)
Moreover the only non-zero Poisson brackets are
\[ \{ x_i, y_j \}_2 = \delta_{ij} \]
\[ \{ x_i, y_j \}_1 = x_i \delta_{ij} \]

**Theorem**

The system \((H_1, \ldots, H_n)\) is separable in \((x_i, y_i)\) \iff \[ \{ H_i, H_j \}_1 = 0 = H_i H_j \]
\[ \forall i, j = 1, \ldots, n. \]

**Different Approach**

Hamiltonian systems admitting a Lax representation with spectral parameter (+ r-matrix formulation)

Separation coordinates are "provided" by the spectral curve

\[ \det(\mu I - L(\lambda)) = 0 \]

\[ L(\lambda) = \text{Lax Matrix} \]

It is possible to find (sometimes)

\[ (x_1, \ldots, x_n, y_1, \ldots, y_n) \] on the phase space
such that:

1. $p_i = (x_i, p_i)$ belongs to the spectral curve $q = i \ldots n$

2. The Hamiltonian system is separable in these coordinates.

- Abel-Jacobi map
- Relative Jacobian
- Family of curves

**Theorem** (Donagi-Freed, Markman-Witten)

1. There is holomorphic symplectic form on $\mathcal{I}$ such that $\pi$ is leg.

$$\mathcal{F}(x_1, \ldots, x_n)$$ coordinates on $B$

- Local, holomorphic $T_{ij} = \frac{\partial^2 \mathcal{F}}{\partial x_i \partial x_j}$
- Holomorphic function on $B$
- Period matrix of $\pi$

2. Equivalently:

$B$ is special-Kähler

- $B$ is Kähler $\omega, I, g$

- $\nabla: \Gamma(TB) \to \Gamma(T^*B \otimes TB)$

- $\nabla \omega = 0$
- $\nabla I = 0$
- $\nabla g = 0$
- Compare these two methods (settings)

- How much we can recover of both of them from a Lagrangian fibration? (Classical Arnold–Liouville I.S.)

**Theorem (Bartocci–M)**

If \( \pi: X \to B \) is a Lagrangian fibration on \( B \) symplectic such that \( \nabla \Theta = 0 \) (\( \Theta \) symplectic form on \( B \), \( \nabla \) linear connec. on \( TB \)) then \( X \) is (locally) Hypercomplex/Hypersymplectic, \( B \) is (locally) Special Kähler
\((X,\omega)\) symplectic manifold \(\dim_{\mathbb{R}}X = 2N\)

\(\pi : X \to B\) fibration \(\pi'(b) = F_b\) cpt connect.

\(\Omega|_{F_b} = 0\)

\((F_b\ are\ tors)\)

For each \(b \in B\)

\(C_b = \{\) invariant vector fields along \(F_b\}\)

\((F_b\ carries\ an\ \mathbb{R}^n\ -\ action)\)

\(C_b \supset \Gamma_b = \{\) invariant vector fields of period \(1\)

\(\Gamma_b \cong H_1(F_b, \mathbb{Z})\)

lattice in \(C_b\)

Define a sheaf on \(B\), \(\Gamma = \{\) period lattice of \(\pi : X \to B\}\)
\( \Gamma \otimes \mathbb{C}^\infty \cong C = \{ \text{Invariant vertical vector fields} \} \)

Vector bundle on \( B \)

\( \pi^* C \cong \text{Vert}(TX) \)

\( U \subset B \) open set \( \quad b \in U \)

\( \{ X_i \} \) basis for \( \Gamma | U \) \( \leadsto \{ \xi_i(b) \} \) basis for \( H_1(C_f, \mathbb{R}) \)

\( d\xi_i = X_i \omega \)

\( \text{Vertical invariant} \quad \text{Hamiltonian} \quad \text{Functions on } \)

\( \text{functions constant along fibers} \quad U \quad (I_2, \ldots, I_n) \)

Can find \( \{ \varphi_i \} \) functions \( \frac{1}{2\pi} \int_{\varphi_i} d\varphi_i = \delta_{1,k} \)

on \( F_b \)

(smooth variable) \wrt \( b \in B \)

Induces: \( (I_2, \ldots, I_n, \varphi_2, \ldots, \varphi_n) \)

Action-Angle coordinates on \( U \subset B \)
\( T^*B \cong C \)
\[ dI_i \rightarrow x_i \]
\[ TB \cong R'_{\pi_B^*} R \otimes C_B^0 \]
\[ \nabla : \Gamma(TB) \rightarrow \Gamma(TB \otimes T^*B) \]
\[ \nabla := 1 \otimes d \]

**Proposition:** \( \nabla \) is flat and torsion free

\[ \nabla^2 = 0 \]

\[ \left\{ \frac{\partial}{\partial x_i} \right\}_{i=1, \ldots, n} \]

provides with a \( \mathbb{Z} \)-basis

for \( \Gamma^* = \text{Hom} (\Gamma, \mathbb{Z}) \)

\[ \mathbb{Z} \]

This gives us an \( \mathbb{Z} \)-basis

for \( R'_{\pi_B^*} \mathbb{R} \):

\[ \Rightarrow \quad \frac{\partial}{\partial x_i} \rightarrow x_i \otimes 1 \]

(This is the \( \cong \))

\[ TB \cong \mathcal{B} R \otimes C_B^0 \]
\[ \nabla \left( \frac{\partial}{\partial x_j} \right) = (1 \wedge d)(x_j \otimes 1) = 0 \]

\[ \implies \quad \text{Connection is torsion free} \]

**Note:**

\[ C \cong T^*B \]

\[ \Gamma \rightarrow \Lambda \quad (\text{Lagrangian covering of } B) \]

loc, freely generated by closed 1-forms

Monodromy of \( \Lambda \) coincide with holonomy of \( \nabla \)
Suppose $\pi : X \rightarrow B$ has a section

**Proposition:** $\triangledown$ induces the following splitting:

$$TX \cong \pi^*(T^*B) \oplus \pi^*(TB)$$

1. $\pi : X \rightarrow B$ has a section

2. $T^*B / \Lambda \cong TX$ (discrete quot.)

   a) $T^*B \rightarrow X$ (proj)
   b) $T^*B \rightarrow X \xrightarrow{\pi} B$
   c) $\pi^*(TX) \cong TT^*B$

3. $\triangledown$ on $T^*B \Rightarrow 0 \rightarrow \text{Vert}(TT^*B) \rightarrow TT^*B \rightarrow \pi^*(TB)$

$$TT^*B \leq \text{Vert}(TT^*B) \oplus \pi^*(TB)$$

4. $\pi : X \rightarrow B$ is lagrangian $\Rightarrow \text{Vert}(TX) \cong \pi^*(TB)$
\[ f^*(TX) \cong f^* \text{Vert}(TX) \oplus (f^* \circ \pi^*)TB \cong (g^* \circ \pi^*)TB \oplus (f^* \circ \pi^*)TB \]

**Consequence:** \( \nabla \) induces a connection on \( TX \)
$T$ is a real Torus

Relative analog.

$\pi: \hat{X} \to \hat{B}$
Torus fibration

$\pi: \hat{X} \to \hat{B}$

$R^1\pi_*\mathbb{R} / \mathbb{R}^1\pi_*\mathbb{Z}$

$\text{If we start with a lag. fibration}$

$\text{Note:}$

$\hat{X} \hat{\nabla} \hat{B}$

$\text{Theorem: } \hat{X} \hat{\nabla} \hat{B}$ is complex manifold

$TX \overset{\nabla}{\nabla} \nabla \cong \pi^*(TB) \oplus \pi^*(TB)$
\[ J : T^* X \to T^* X \]
\[ (\alpha, \beta) \mapsto (-\beta, \alpha) \]

is almost complex. Integrability follows from compatibility with \( \nabla \)
(\text{flat and torsion free}).

Local coordinates: \( (I_1, \ldots, I_n, \Phi_1, \ldots, \Phi_n) \)

\( \{ \Phi_i \} \) are dual coordinates to \( \{ \xi_i \} \)

\[ 2k = Jk + i \xi_k \quad \text{are holomorphic} \]

w.r.t. \( J \).
**NOTE:**

If \( X \xrightarrow{\pi} B \) has a section

\[ X \simeq \hat{X} \] (as real Torus Fib)

② the connection on \( X \) induced by the one on \( T^B \) will coincide with the one coming from

\[ T\hat{X} \simeq \pi^*T^B \oplus \hat{\pi}^*T\hat{B} \]

③ \( \{ \Phi_i \} \) and \( \{ \hat{\Phi}_i \} \) under

The isomorphism \( X \simeq \hat{X} \)
We will work locally

1) Suppose $B$ is symplectic, $\Omega$ symplectic form

2) We suppose $\sigma: B \to X$ (section) or $\delta\omega: B \to X$
   
   Lagrangian section

   (Locally it is always true because Arnold-Liouville Th.)

3) $\nabla \Omega = 0$
\[ T^*B \cong TB \]

\[ 0 \rightarrow \text{Vert}(TM) \rightarrow TM \rightarrow \pi^*(TB) \rightarrow 0 \]

\[ TM \cong \pi^*(TB) \oplus \text{Vert}(TM) \cong \pi^*(TB) \oplus \pi^*(TB) \]

\[
\Rightarrow \quad X = (-\Omega') \oplus \Omega \quad \text{symplectic form on } X
\]

\[
\text{Vertical}
\]

\[
\text{Hor.}
\]

Since \( \nabla \Omega = 0 \implies \Gamma = (x_1, \ldots, x_n, y_1, \ldots, y_m) \)

are flat symplectic

\[ \Omega = dx_1 \wedge dy_1 \quad X = -dp_1 \wedge dq + dx_1 \wedge dy_1 \]

\[ \Psi = (p_1, \ldots, p_n, q_1, \ldots, q_m) \]

(are vertical coordinates in \( X \))
Lemma: The symplectic form $\omega$ and $\omega$ (on $X$) can be written as:

1. $\omega = dq \wedge dp - dp \wedge dx + dq \wedge dy$

$$(p,q) = \{q_i; \dot{q}_i\} \quad p_i = \Phi_i \quad i = 1, \ldots, n$$

$q_i = \Phi_i + \chi_i \quad i = 1, \ldots, n$

2. $X = -dp \wedge dq + dx \wedge dy$

On $X$:

$$J_\omega = dx \otimes \frac{\partial}{\partial p} - dp \otimes \frac{\partial}{\partial x} + dy \otimes \frac{\partial}{\partial y} - dq \otimes \frac{\partial}{\partial q}$$

$$J_x = -dp \otimes \frac{\partial}{\partial q} + dq \otimes \frac{\partial}{\partial p} + dx \otimes \frac{\partial}{\partial y} - dy \otimes \frac{\partial}{\partial x}$$

(which is defined only locally)

$$(x,y,p,q)$$

$$(I_i; \Phi_i; \chi_i; \dot{q}_i)$$
Define: \[ K = J_w \circ J_x \]

**Proposition**

1. \( K \) is complex structure on \( \mathring{T_X} \)
2. holomorphic coordinates \((z_2, \ldots, z_n, \beta_2, \ldots, \beta_n)\)

\[
dz_2 = dx + i \, dz_2 \
d\beta_i = dP_i + i \, dy_i
\]

On \( X \) we have:

\[
\Omega = dP_2 \wedge dx + dy \wedge dy
\]

\[
K = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial P_2}{\partial x} \frac{\partial P_2}{\partial y} + dy_2 \wedge \frac{\partial P_2}{\partial P_i} \\
= : J_0
\]
Theorem: \( X \supset B \) (real) leg. fib \( \rightarrow \)
\( (B, \pi) \quad \forall \pi = 0 \)

then the choice of a section of \( \pi \) over \( U \subseteq B \) defines hypersymplectic structure on \( X|_U \) and hypercomplex structure on \( \mathbb{R}|_U \)

\[ X|_U \cong \hat{X}|_U \]
\[ (x_i, y, p_{1,4}) \sim (x_i, y, p_{1,4}) \]

\[ \Phi \]
\[ J_w = dx \otimes \frac{\partial}{\partial p} - dp \otimes \frac{\partial}{\partial x} + dy \otimes \frac{\partial}{\partial q} - dq \otimes \frac{\partial}{\partial y} \]

Induces \((\mathbf{w}, w) = (x + ip, y + iq)\) local holomorphic coordinates \((\text{on } X)\)

Fix \(\phi: B \rightarrow X\) Lagrangian section (locomally it is pos.)

\(\phi(B) \subseteq X\) is complex submanifold \(w.r.t \text{ to } J_w\)

Moreover: \(J_w \bigg|_{\phi(B)}\) induces \(\phi(B) \subseteq \mathbb{C}^n\)

\(I \in \text{End}(TB)\)

\(I^2 = -\text{Id}\)

Local coordinates: \(I = -(dp \otimes \frac{\partial}{\partial x} + dq \otimes \frac{\partial}{\partial y})\)
Lemma: \[ \nabla I = 0 \]

\((x, i y)\) are flat coordinates w.r.t. \(\nabla\).

Theorem: \((\beta, \Omega, I, \nabla)\) is pseudo special Kähler.