Evaluating Kinematical Factors of Pure Spinor Scattering Amplitudes

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5. What is left to do
What do I want to do?

- Compute the kinematical factors of amplitudes obtained with the pure spinor formalism
  - PRL 96 (2006), 011602 (Berkovits, C.M.)
  - JHEP 0601 (2006), 075 (C.M.)
  - JHEP 0611 (2006), 079 (Berkovits, C.M.)

- Check if they agree with RNS and GS results
  - For example, the 4-point 1-loop kinematical factor:
    \[
    \langle (\lambda A^1)(\lambda \gamma^m W^2)(\lambda \gamma^n W^3) F^4_{mn} \rangle + \text{perm}(234) = t_8 F^4 + \text{fermions}
    \]
  - Or the 4-point amplitude at 2-loops
    \[
    \langle (\lambda \gamma^{mnpqr} \lambda) F^1_{mn} F^2_{pq} F^3_{rs} (\lambda \gamma^s W^4) \rangle + \text{perm}(1234) = (t - u) t_8 F^4 + \ldots
    \]
  - Or the gauge variation of the 6-point amplitude at 1-loop
    \[
    \langle (\lambda \gamma^m W)(\lambda \gamma^n W)(\lambda \gamma^p W) (W \gamma_{mnp} W) \rangle = \epsilon_{10} F^5
    \]
What do I want to do?

- Or how to prove the following interesting identity

\[
\langle (\lambda \gamma^r \gamma^{m_1 n_1} \theta)(\lambda \gamma^s \gamma^{m_2 n_2} \theta)(\lambda \gamma^t \gamma^{m_3 n_3} \theta)(\theta \gamma^m \gamma^n \gamma^{rst} \gamma^{m_4 n_4} \theta) \rangle =
\]

\[
= -\frac{2}{45} \left( \eta^{mn} t_8^{m_1 n_1 m_2 n_2 m_3 n_3 m_4 n_4} - \frac{1}{2} \epsilon_{10}^{mn m_1 n_1 m_2 n_2 m_3 n_3 m_4 n_4} \right)
\]

- Find tricks and shortcuts to compute general scattering amplitudes

- Note: Compact notation \( t_8 F^4 \) means

\[
t_8 F^4 \equiv 4(F^1 F^2 F^3 F^4) + 4(F^1 F^3 F^2 F^4) + 4(F^1 F^2 F^4 F^3)
\]

\[
- (F^1 F^2)(F^3 F^4) - (F^1 F^3)(F^2 F^4) - (F^1 F^2)(F^4 F^3)
\]
The pure spinor formalism is a CFT based on the following action

Action (Minimal Pure Spinor Formalism)

\[ S = \int d^2 z \left( \frac{1}{2} \partial X^m \overline{\partial} X_m + p_\alpha \overline{\partial} \theta^\alpha - w_\alpha \overline{\partial} \lambda^\alpha \right) \]

With a bosonic pure spinor \( \lambda^\alpha \)

Constraints

\[ (\lambda \gamma^m \lambda) = 0 \]
Some important definitions for amplitude computations:

- **Lorentz current**
  \[
  N^{mn} = \frac{\alpha'}{4} (\mathcal{W} \gamma^{mn} \lambda)
  \]

- **Supersymmetric momentum**
  \[
  \Pi^m = \partial X^m + \frac{1}{2} (\theta \gamma^m \partial \theta)
  \]

- **Supersymmetric derivative**
  \[
  D_\alpha = \frac{\partial}{\partial \theta^\alpha} + \frac{1}{2} (\theta \gamma^m)_\alpha \partial_m
  \]
Supersymmetric Green-Schwarz constraint

\[ d_{\alpha} = \frac{\alpha'}{2} p_{\alpha} - \frac{1}{2} (\gamma^m \theta)_\alpha \partial X_m - \frac{1}{8} (\gamma^m \theta)_\alpha (\theta \gamma_m \partial \theta) \]
Pure Spinor Formalism

### Relevant OPE's

\[ X^m(z, \bar{z})X^n(w, \bar{w}) \rightarrow -\frac{1}{2} \eta^{mn} \ln |z - w|^2 \]

\[ N^{mn}(z) \lambda^\alpha(y) \rightarrow \frac{\alpha' \left( \gamma^{mn} \lambda \right)^\alpha}{4 \frac{1}{z - y}} \]

\[ d_\alpha(z)V(y, \theta) \rightarrow \frac{D_\alpha V(y, \theta)}{z - y} \]

\[ \Pi^m(z)V(y, \theta) \rightarrow \frac{\partial^m V(y, \theta)}{z - y} \]
Issues of RNS and GS not present

Space-time SUSY
The pure spinor formalism has manifest space-time supersymmetry

Covariant BRST Quantization

\[ Q_{\text{BRST}} = \int \lambda^\alpha d_\alpha \]
Prescription for Scattering Amplitudes

- **Massless Vertex Operators:**
  - Unintegrated
    \[ V = \lambda^\alpha A_\alpha(X, \theta) \]
  - Integrated
    \[ U = \int dz \left( \partial \theta^\alpha A_\alpha + A_m \Pi^m + d_\alpha W^\alpha + \frac{1}{2} N^{mn} F_{mn} \right) \]

- Where \( A_\alpha(x, \theta), A_m(x, \theta), W^\alpha(x, \theta) \) and \( F_{mn}(x, \theta) \) are the SYM superfields.
The prescription for tree-level amplitudes is given by

\[ A_N = \langle V_1(z_1) V_2(z_2) V_3(z_3) \int dz_4 U_4(z_4) \ldots \int dz_N U_N(z_N) \rangle \]

Computation proceeds as usual in a CFT
- Use OPE’s to integrate out conformal weight 1 variables
- Then integrate out zero-modes
For our purposes now, integration over $\lambda^\alpha$ and $\theta^\alpha$ zero-modes is done with the rule

$$\langle (\lambda \gamma^m \theta)(\lambda \gamma^n \theta)(\lambda \gamma^p \theta)(\theta \gamma_{mnp} \theta) \rangle = 1$$
The computation of scattering amplitudes gives rise to pure spinor superspace expressions.

Compact way of writing the full amplitude:
- Contain all possible contributions of fermionic and bosonic external states.

To compare results with RNS/GS one has to express these pure spinor expressions in terms of polarization and momenta.

This is now a solved problem:
- Systematic procedure to evaluate pure spinor superspace expressions in components.
- I have made Mathematica functions that make this job.
Four gravitons at tree-level

\[ \mathcal{A} = \langle V_1(z_1, \bar{z}_1) V_2(z_2, \bar{z}_2) V_3(z_3, \bar{z}_3) \int_{\mathbb{C}} d^2z U_4(z, \bar{z}) \rangle \]

where \( V^i(z, \bar{z}) = V^i(z) \otimes \tilde{V}^i(\bar{z}) e^{ik \cdot X} \) and \( U(z, \bar{z}) = U(z) \otimes \tilde{U}(\bar{z}) e^{ik \cdot X} \)
Previous computation (Policastro, Tsimpis 2006) were done in a way that hid the simplicity of the result. Cancellations were overlooked and no simple pure spinor expression was written down for the kinematical factor.

- We have to compute

\[ \langle (\lambda A^1)(z_1)(\lambda A^2)(z_2)(\lambda A^3)(z_3) \int d^2z (\prod^m A^4_m + (dW^4) + \frac{1}{2} N^{mn} F_{mn}) \rangle \]

⊗ (right-moving part)

- SL(2,C) invariance allows the fixing \( z_1 = 0, z_2 = 1 \) and \( z_3 \to \infty \)
\( \Pi^m A^4_m \) term of integrated vertex contribute only with

\[
\langle (\lambda A^1)(\lambda A^2)(\lambda A^3) A^4_m \Pi^m : e^{i k_1} X :: e^{i k_2} X : e^{i k_3} X : e^{i k_4} X : \rangle =
\]

\[
= \sum_{i=1}^{2} \alpha' \frac{i k_i^m}{2} \frac{1}{z_i - z_4} \langle (\lambda A^1)(\lambda A^2)(\lambda A^3) A^4_m \rangle \otimes \Pi(Z_{ij})
\]
One can use some identities to simplify result of other OPE’s

Delay as long as possible explicit evaluation of pure spinor integrals

**Lemma**

One can show the OPE identity

\[
\langle (\lambda A^1)(\lambda A^2)(\lambda A^3)((dW^4) + \frac{1}{2} N^{mn} F_{mn}) \rangle = \\
+ \frac{\alpha'}{2(z_1 - z_4)} \langle A^1_m(\lambda A^2)(\lambda A^3)(\lambda \gamma^m W^4) \rangle - (1 \leftrightarrow 2) + (1 \leftrightarrow 3)
\]
We organize the computation as

\[ \mathcal{A} = \text{const} \int d^2 z_4 \left( \frac{F_1}{z_1 - z_4} + \frac{F_2}{z_2 - z_4} \right) \otimes \left( \frac{\tilde{F}_1}{\tilde{z}_1 - \tilde{z}_4} + \frac{\tilde{F}_2}{\tilde{z}_2 - \tilde{z}_4} \right) \]

\[ \cdot |z_4|^{-\alpha'/2} |1 - z_4|^{-\alpha' u/2} \]

where

\[ F_1 = i k_m \langle (\lambda A^1)(\lambda A^2)(\lambda A^3) A_m^4 \rangle + \langle A_m^1(\lambda A^2)(\lambda A^3)(\gamma^m W^4) \rangle \]

\[ F_2 = i k_m \langle (\lambda A^1)(\lambda A^2)(\lambda A^3) A_m^4 \rangle - \langle (\lambda A^1) A_m^2(\lambda A^3)(\gamma^m W^4) \rangle \]
Using the general formula

\[ \int d^2z z^A (1 - z)^B \bar{z}^\tilde{A} (1 - \bar{z})^\tilde{B} = \frac{2\pi \Gamma(1 + A)\Gamma(1 + B)}{\Gamma(2 + A + B)} \cdot \frac{\Gamma(-1 - \tilde{A} - \tilde{B})}{\Gamma(-\tilde{A})\Gamma(-\tilde{B})} \]

we get

\[ \mathcal{A} = K\tilde{K} \frac{\Gamma(-\alpha'/t/4)\Gamma(-\alpha'/u/4)\Gamma(-\alpha'/s/4)}{\Gamma(1 + \alpha'/s/4)\Gamma(1 + \alpha'/t/4)\Gamma(1 + \alpha'/u/4)} \]

where

\[ K = uF_1 - tF_2 \quad \tilde{K} = u\tilde{F}_1 - t\tilde{F}_2 \]
Tree-level 4-graviton result

$\mathcal{A} = K \otimes \tilde{K} \frac{\Gamma(-s/4)\Gamma(-t/4)\Gamma(-u/4)}{\Gamma(1+s/4)\Gamma(1+t/4)\Gamma(1+u/4)}$

where the kinematical factor is given by

$K = \langle \partial^n(\lambda A^1)\partial^m(\lambda A^2)(\lambda A^3)\mathcal{F}^4_{mn} \rangle$

$+ \langle (\partial_p A^1_m)(\lambda A^2)\partial^p(\lambda A^3)(\lambda \gamma^m W^4) \rangle$

$+ \langle (\lambda A^1)(\partial_p A^2_m)\partial^p(\lambda A^3)(\lambda \gamma^m W^4) \rangle$
Massless 4-point one-loop amplitude

\[ A_N = \langle N \left( \int \mu \cdot b \right) V_1(z_1) \int U_2 \int U_3 \int U_4 \rangle \]
This amplitude was computed with the minimal pure spinor formalism (Berkovits 2004) and shown to agree with the RNS and GS results (C.M. 2005).

Computed also in the non-minimal pure spinor formalism (Berkovits 2005, Berkovits & C.M. 2006)
4-gravitons interaction at one-loop order

\[ \mathcal{A} = K \otimes \tilde{K} \int \frac{d^2 \tau}{(\text{Im} \tau)^2} F(\tau) \]

- Minimal Pure Spinor Formalism

\[ K_{\text{one-loop}} = \langle (\lambda A)(\lambda \gamma^m W)(\lambda \gamma^n W) F_{mn} \rangle \]

- Now one has to show that \( K_{\text{one-loop}} \) is proportional to \( t_8 F^4 \)
- How to do that?
4-gravitons interaction at one-loop order

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Evaluating Pure Spinor Superspace Expressions

\[ \int \,[d\lambda][d\bar{\lambda}]\lambda^\alpha \lambda^\beta \lambda^\gamma \lambda^\delta \bar{\lambda}_\rho \bar{\lambda}_\sigma \bar{\lambda}_\tau \bar{\lambda}_\omega \]

\[ \langle \lambda \bar{\lambda} \rangle \langle \lambda^3 \theta^5 \rangle \]

???
Evaluating Pure Spinor Superspace Expressions

- Pure spinor superspace expressions are compact and elegant
- However, until the Pure Spinor Formalism becomes the *de facto* standard superstring formalism, one needs to check the results in components
- Straightforward to do with the $(\lambda^3 \theta^5)$ rule

\[ \langle (\lambda \gamma^m \theta)(\lambda \gamma^n \theta)(\lambda \gamma^p \theta)(\theta \gamma_{mnp} \theta) \rangle = 1 \]
Suppose one wants to compute the 1-loop pure spinor superspace integral

\[ \langle (\lambda A)(\lambda \gamma^m W)(\lambda \gamma^n W)F_{mn} \rangle \]

We first expand superfields in \( \theta \)'s as follows
Evaluating Pure Spinor Superspace Expressions

### SYM Superfields $\theta$-Expansion

\[ A_\alpha(x, \theta) = \frac{1}{2} a_m (\gamma^m \theta) \alpha - \frac{1}{3} (\xi \gamma_m \theta) (\gamma^m \theta) \alpha - \frac{1}{32} F_{mn} (\gamma^p \theta) \alpha (\theta \gamma^{mnp} \theta) + \ldots \]

\[ A_m(x, \theta) = a_m - (\xi \gamma_m \theta) - \frac{1}{8} (\theta \gamma_m \gamma^{pq} \theta) F_{pq} + \frac{1}{12} (\theta \gamma_m \gamma^{pq} \theta) (\partial_p \xi \gamma_q \theta) + \ldots \]

\[ W^\alpha(x, \theta) = \xi^\alpha - \frac{1}{4} (\gamma^{mn} \theta)^\alpha F_{mn} + \frac{1}{4} (\gamma^{mn} \theta)^\alpha (\partial_m \xi \gamma_n \theta) \]

\[ + \frac{1}{48} (\gamma^{mn} \theta)^\alpha (\theta \gamma_n \gamma^{pq} \theta) \partial_m F_{pq} + \ldots \]

\[ F_{mn}(x, \theta) = F_{mn} - 2 (\partial_m \xi \gamma_n \theta) + \frac{1}{4} (\theta \gamma_m \gamma^{pq} \theta) \partial_n F_{pq} + \ldots, \]
Remember that correlator must have 5 $\theta$’s to be non-zero
If we want the bosonic contribution we distribute $\theta$’s as follows

<table>
<thead>
<tr>
<th>$A_{\alpha}(\theta)$</th>
<th>$W^\alpha(\theta)$</th>
<th>$W^\alpha(\theta)$</th>
<th>$\mathcal{F}_{mn}(\theta)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
Sidenote

Previous computation (Anguelova, Grassi, Vanhove 2004) was wrong. Omitted first three lines of above table.

- In JHEP 0601 (2006) (C.M.) it was shown that to get the right result one also has to include the first three lines.
- In JHEP 0705 (2007) (C. Stahn) the fermionic contributions were also computed.
Evaluating Pure Spinor Superspace Expressions

Considering all lines of the table we get

\[ K_{1}^{NS} = + \frac{15}{64} F_{mn}^{1} F_{pq}^{2} F_{rs}^{3} F_{tu}^{4} \langle \lambda \gamma^{[t} \gamma^{pq} \theta \rangle \langle \lambda \gamma^{u]} \gamma^{rs} \theta \rangle \langle \lambda \gamma^{a} \theta \rangle \langle \theta \gamma^{mna} \theta \rangle \rangle + \]

\[ + \frac{15}{16} (k_{m} e_{n}^{1}) F_{pq}^{2} F_{rs}^{3} F_{tu}^{4} \langle \lambda \gamma^{[m} \gamma^{pq} \theta \rangle \langle \lambda \gamma^{a]} \gamma^{rs} \theta \rangle \langle \lambda \gamma^{n} \theta \rangle \langle \theta \gamma^{a} \gamma^{tu} \theta \rangle \rangle + \]

\[ + \frac{5}{16} (k_{m} e_{n}^{1}) F_{pq}^{2} F_{rs}^{3} F_{tu}^{4} \langle \lambda \gamma^{[t} \gamma^{ma} \theta \rangle \langle \lambda \gamma^{u]} \gamma^{rs} \theta \rangle \langle \lambda \gamma^{n} \theta \rangle \langle \theta \gamma^{a} \gamma^{pq} \theta \rangle \rangle + \]

\[ + \frac{5}{16} (k_{m} e_{n}^{1}) F_{pq}^{2} F_{rs}^{3} F_{tu}^{4} \langle \lambda \gamma^{[t} \gamma^{pq} \theta \rangle \langle \lambda \gamma^{u]} \gamma^{ma} \theta \rangle \langle \lambda \gamma^{n} \theta \rangle \langle \theta \gamma^{a} \gamma^{rs} \theta \rangle \rangle. \]
Practical Question

How do we compute \( \langle (\lambda \gamma^t|\gamma^{pq}\theta)(\lambda \gamma^u|\gamma^rs\theta)(\lambda \gamma^a\theta)(\theta \gamma^{mna}\theta) \rangle \) or
\[
\langle (\lambda \gamma^m\gamma^{m_1n_1}\theta)(\lambda \gamma^n\gamma^{m_2n_2}\theta)(\lambda \gamma^p\gamma^{m_3n_3}\theta)(\theta \gamma^{m_4n_4}\gamma_{mnp}\gamma^{m_5n_5}\theta) \rangle \\
\langle (\lambda \gamma^m\theta)(\lambda \gamma^a\gamma^{m_1n_1}\theta)(\lambda \gamma^{bcn}\gamma^{m_2n_2}\theta)(\theta \gamma^{m_3n_3}\gamma_{abc}\gamma^{m_4n_4}\theta) \rangle
\]
in general?

- One has to relate a general pure spinor superspace expression to
  \( \langle (\lambda \gamma^m\theta)(\lambda \gamma^n\theta)(\lambda \gamma^p\theta)(\theta \gamma_{mnp}\theta) \rangle \)
- One can always do that by symmetry arguments
- Example:
  \[
  \langle (\lambda \gamma^m\theta)(\lambda \gamma^n\theta)(\lambda \gamma^p\theta)(\theta \gamma_{ijk}\theta) \rangle = \frac{1}{120} \delta_{ijk}^{mnp}
  \]
Identities

\[ (\lambda \gamma^m \gamma^{np} \theta) = (\lambda \gamma^{mnp} \theta) + \eta^{mn}(\lambda \gamma^p \theta) - \eta^{mp}(\lambda \gamma^n \theta) \]

\[ (\lambda \gamma^{abc} \gamma^{de} \theta) = + (\lambda \gamma^{abcde} \theta) - 2\delta_{de}^{bc}(\lambda \gamma^a \theta) + 2\delta_{de}^{ac}(\lambda \gamma^b \theta) - 2\delta_{de}^{ab}(\lambda \gamma^c \theta) \]

\[-\delta_e^{c}(\lambda \gamma^{abd} \theta) + \delta_d^{c}(\lambda \gamma^{abe} \theta) + \delta_e^{b}(\lambda \gamma^{acd} \theta) - \delta_d^{b}(\lambda \gamma^{ace} \theta) \]

\[ -\delta_e^{a}(\lambda \gamma^{bcd} \theta) + \delta_d^{a}(\lambda \gamma^{bce} \theta) \]

\[ (\theta \gamma^{m_4 n_4} \gamma_{mnp} \gamma^{m_5 n_5} \theta) = G_{mnp r_1 r_2 r_3}^{m_4 n_4 m_5 n_5} (\theta \gamma^{r_1 r_2 r_3} \theta) \text{ where} \]

\[ G_{mnp r_1 r_2 r_3}^{m_4 n_4 m_5 n_5} = \frac{1}{6} \epsilon_{mm_4 m_5 nn_4 n_5 pr_1 r_2 r_3} - 24\delta_{n_4 n_5}^{np} \delta_{r_1 r_2 r_3}^{mm_4 m_5} + 12\delta_{n_4 p}^{m_5 n_5} \delta_{r_1 r_2 r_3}^{mm_4 n} \]

\[-6\delta_{n_4 n_5}^{m_5 n_5} \delta_{r_1 r_2 r_3}^{mm_4 n_4} + 12\delta_{n_5 p}^{m_4 n_4} \delta_{r_1 r_2 r_3}^{mm_5 n} - 6\delta_{n_5 p}^{m_4 n_4} \delta_{r_1 r_2 r_3}^{mm_5 n_5} - 2\delta_{m_5 n_5}^{m_4 n_4} \delta_{r_1 r_2 r_3}^{m_5 n_5} \]

\[ + [mnp] + [m_4 n_4] + [m_5 n_5], \]
\begin{align*}
(\theta \gamma^{abc} \gamma^{mn}\theta) &= (\theta \gamma^{r_1r_2r_3}\theta)K^{abc{r_1r_2r_3}}_{r_1r_2r_3} \text{ where} \\
K^{abc{r_1r_2r_3}}_{r_1r_2r_3} &= -\eta^{cn}\delta^{abm}_{r_1r_2r_3} + \eta^{cm}\delta^{abn}_{r_1r_2r_3} + \eta^{bn}\delta^{acm}_{r_1r_2r_3} \\
&- \eta^{bm}\delta^{acn}_{r_1r_2r_3} - \eta^{an}\delta^{bcm}_{r_1r_2r_3} + \eta^{am}\delta^{bcn}_{r_1r_2r_3} \\
\end{align*}

\begin{align*}
(\gamma^{mnp})_{\alpha\beta}(\gamma^{mnp})^{\gamma\delta} &= 48\left(\delta^{\gamma\delta}_{\alpha} - \delta^{\gamma\delta}_{\beta}\right), \quad (\lambda\gamma^{m}\psi)(\lambda\gamma^{m}\xi) = 0 \quad \forall \psi^{\alpha}, \xi^{\alpha} \\
(\lambda\gamma^{mnpr}\lambda)(\lambda\gamma^{mn}\theta) &= 0, \quad (\lambda\gamma^{mnqr}\lambda)(\lambda\gamma^{m}\theta) = 0 \\
The identities below are necessary to get recursion relations between different pure spinor superspace identities
\begin{align*}
(\lambda\gamma^{amn}\theta)(\lambda\gamma^{a}\theta) &= 2(\lambda\gamma^{m}\theta)(\lambda\gamma^{n}\theta) \\
(\lambda\gamma^{abm}\theta)(\lambda\gamma^{abn}\theta) &= -4(\lambda\gamma^{m}\theta)(\lambda\gamma^{n}\theta)
\end{align*}
\]
\[(\lambda \gamma^{mabcn} \lambda)(\theta \gamma_{abc} \theta) = 96(\lambda \gamma^m \theta)(\lambda \gamma^n \theta),\]
\[(\lambda \gamma^{abc} \lambda)(\lambda \gamma_{abc} \theta) = -36(\lambda \gamma^m \theta)(\lambda \gamma^n \theta),\]
\[(\lambda \gamma^a \gamma^{bcmn} \theta)(\lambda \gamma_{abc} \theta) = -28(\lambda \gamma^m \theta)(\lambda \gamma^n \theta),\]
\[(\lambda \gamma^{abc} \theta)(\lambda \gamma^{ade} \theta) = -(\lambda \gamma^{cde} \theta)(\lambda \gamma^b \theta) + (\lambda \gamma^{bde} \theta)(\lambda \gamma^c \theta)
   + (\lambda \gamma^{bce} \theta)(\lambda \gamma^d \theta) - (\lambda \gamma^{bcd} \theta)(\lambda \gamma^e \theta)
   - \eta^{ce}(\lambda \gamma^b \theta)(\lambda \gamma^d \theta) + \eta^{cd}(\lambda \gamma^b \theta)(\lambda \gamma^e \theta)
   + \eta^{be}(\lambda \gamma^c \theta)(\lambda \gamma^d \theta) - \eta^{bd}(\lambda \gamma^c \theta)(\lambda \gamma^e \theta)\]
And many many more...
Evaluating Pure Spinor Superspace Expressions

Pure Spinor Superspace Identities

\[
\langle (\lambda \gamma^m \theta)(\lambda \gamma^n \theta)(\lambda \gamma^p \theta)(\theta \gamma_{ijk} \theta) \rangle = \frac{1}{120} \delta_{ijk}^{mnp}
\]

\[
\langle (\lambda \gamma^{mnp} \theta)(\lambda \gamma q \theta)(\lambda \gamma t \theta)(\theta \gamma_{ijk} \theta) \rangle = \frac{1}{70} \delta_{[m \eta t][i \eta^j \eta^k]}^{[q \eta]} \delta_p^{\ell}
\]

\[
\langle (\lambda \gamma_t \theta)(\lambda \gamma^{mnp} \theta)(\lambda \gamma^{qrs} \theta)(\theta \gamma_{ijk} \theta) \rangle = \frac{1}{8400} \epsilon_{ijkmnpqrs t}
\]

\[
+ \frac{1}{140} \left[ \delta_t^{[m \eta^p]} \delta_{[i \eta^q]}^{\ell} \delta_{\ell j}^{[s \eta^k]} - \delta_t^{[q \eta^s]} \delta_{[i \eta^t]}^{\ell} \delta_{\ell j}^{[m \eta^k]} \right]
\]

\[
- \frac{1}{280} \left[ \eta_t^{[i \eta^q]} \delta_{\ell j}^{[s \eta^k]} \delta_{\ell v}^{[m \eta^p]} - \eta_t^{[i \eta^q]} \delta_{\ell j}^{[m \eta^p]} \delta_{\ell v}^{[s \eta^k]} \right]
\]
It is straightforward to compute pure spinor superspace expressions in components (although tedious):

1. Substitute SYM superfields by their theta expansions
2. Use above PS superspace identities (and a lot more!)
3. Write everything in terms of polarization \((e^m_i, \xi^\alpha)\) and momenta

Doing that we finally get

\[
\langle (\lambda A^1)(\lambda \gamma^m W^2)(\lambda \gamma^n W^3) F^4_{mn} \rangle + \text{perm} = t_8 F^4 + \text{fermions}
\]

**PS survived**

Pure spinor formalism is equivalent to RNS/GS at 1-loop order
Evaluating Pure Spinor Superspace Expressions

- It is straightforward to compute pure spinor superspace expressions in components (although tedious):
  1. Substitute SYM superfields by their theta expansions
  2. Use above PS superspace identities (and a lot more!)
  3. Write everything in terms of polarization \( e_i^m, \xi^\alpha \) and momenta

- Doing that we finally get

\[
\langle (\lambda A^1)(\lambda \gamma^m W^2)(\lambda \gamma^n W^3)F_{mn}^4 \rangle + \text{perm} = t_8 F^4 + \text{fermions}
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Pure spinor formalism is equivalent to RNS/GS at 1-loop order
Massless 4-point two-loop amplitude

$A_N = \langle \mathcal{N} \left( \int \mu \cdot b \right) \left( \int \mu \cdot b \right) \left( \int \mu \cdot b \right) \int U_2 \int U_3 \int U_4 \rangle$
4 gravitons at two-loop order

\[ \mathcal{A} = K \otimes \tilde{K} \int d^2 \Omega_{11} d^2 \Omega_{12} d^2 \Omega_{22} \prod_{i=1}^{4} \int d^2 z_i \frac{\exp \left( -\sum_{i,j=1}^{4} k_i \cdot k_j G(z_i, z_j) \right)}{\left( \det \text{Im}\Omega \right)^5} \]

where

\[ K_{\text{two-loop}} = \langle (\lambda \gamma^{mnpqr} \lambda) \mathcal{F}_{mn}^{1} \mathcal{F}_{pq}^{2} \mathcal{F}_{rs}^{3} (\lambda \gamma^{s} W^{4}) \rangle \Delta(z_1, z_3) \Delta(z_2, z_4) + \text{perm}(1234) \]
Using the above procedure it was shown (Berkovits, C.M., 2005) that

\[ \langle (\lambda \gamma^{mnpqr} \lambda) F^1_{mn} F^2_{pq} F^3_{rs} (\lambda \gamma^s W^4) \rangle \Delta(z_1, z_3) \Delta(z_2, z_4) \]

\[ + \text{perm}(1234) = (t - u) t_8 F^4 \Delta(z_1, z_2) \Delta(z_3, z_4) + \ldots \]

Fermionic terms were also computed (C. Stahn, 2007)

Comparing it with the RNS result (D’Hoker, Phong, 2005)...

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Gauge Variation of Massless 6-point one-loop amplitude

\[ \delta A_N = \langle \mathcal{N} \left( \int \mu \cdot b \right) Q_{BRST} \Omega \int U_2 \int U_3 \int U_4 \int U_5 \int U_6 \rangle \]
Gauge Variation of Massless 6-point one-loop amplitude

- Gauge variation of unintegrated vertex is $\delta (\lambda^\alpha A_{\alpha}) = Q_{BRST} \Omega$

- Computed in the non-minimal pure spinor formalism (Berkovits & C.M. 2006)

**Pure Spinor Superspace Result**

$$\delta A = K_{anom} \times \text{moduli space part}$$

where

$$K_{anom} = \langle (\lambda \gamma^m W)(\lambda \gamma^n W)(\lambda \gamma^p W)(W \gamma_{mnp} W) \rangle$$

$$= \epsilon_{10} F^5$$
Four gravitons at tree-level

\[ K_{\text{tree}} = (t_8 \cdot F^4) \]

Four gravitons at one-loop

\[ K_{\text{one-loop}} = (t_8 \cdot F^4) \]

Four gravitons at two-loop

\[ K_{\text{two-loop}} = (t_8 \cdot F^4) [(t - u)\Delta(z_1, z_2)\Delta(z_3, z_4) + \text{permutations}] \]

Anomaly kinematic factor

\[ K_{\text{anom}} = (\epsilon_{10} \cdot F^5) \]
Results are OK

- Everything done so far agrees with standard RNS and GS results
  - General proof for tree-level equivalence (Berkovits, Valillo 2000)
  - Equivalence at one- and two-loop level by explicit computation (Berkovits, C.M. 2005/2006)
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Compute the coefficients and check unitarity (work in progress)
Compute higher-point amplitudes
Study the properties of pure spinor superspace integrals
THE END