Yangians In Deformed Super Yang-Mills Theories

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Outline

1. Background
2. Algebra
3. Hamiltonian
4. Deformed Hamiltonian
5. Twisted Coproducts and Broken Symmetry
\( \mathcal{N} = 4, \ D = 4 \) Super Yang-Mills has PSU(2,2|4) superalgebra.

A Yangian extension exists in the planar limit of the SU(N) gauge group.

Marginal deformations have been used to deform to \( \mathcal{N} = 1, \ D = 4 \) SYM theories, SU(2, 2|1) × U(1) × U(1).

Deformations (beta and twists) have been shown to maintain integrability.

Twisted theories have a non-standard coproduct.
Marginal Deformation

We break the $\mathcal{N} = 4$ to a $\mathcal{N} = 1$ superconformal theory by the addition of the marginal deformation with the superpotential

$$\mathcal{W} = ih \text{Tr} \left( e^{i\pi\beta} \Phi_1 \Phi_2 \Phi_3 - e^{-i\pi\beta} \Phi_1 \Phi_3 \Phi_2 \right) + \frac{ih'}{3} \text{Tr} \left( \Phi_1^3 + \Phi_2^3 + \Phi_3^3 \right).$$

For an exact marginal deformation,

$$|h|^2 \left( 1 + \frac{1}{N} \left( e^{i\pi\beta} - e^{-i\pi\beta} \right)^2 \right) + |h'|^2 \frac{N^2 - 4}{2N^2} = g^2.$$

We left $h' = 0$ and $\beta$ real in the large $N$ limit. Then $h = g$. 
The Deformed Lagrangian

The Lagrangian of a deformed $\mathcal{N} = 4, D = 4$ SYM:

$$\mathcal{L} = \frac{1}{g^2} \text{Tr} \left( \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + (\mathcal{D}^\mu \Phi^i) (\mathcal{D}_\mu \Phi_i) - \frac{1}{2} [\Phi_i, \Phi_j] C_{ij} [\Phi^i, \Phi^j] C_{ij} \right. $$

$$+ \frac{1}{4} [\Phi_i, \Phi^i] [\Phi_j, \Phi^j] + \lambda_A \sigma^\mu \mathcal{D}_\mu \bar{\lambda}^A - i[\lambda_4, \lambda_i] B_{4i} \Phi^i $$

$$+ i[\bar{\lambda}^4, \bar{\lambda}^i] B_{4i} \Phi_i + \frac{i}{2} \epsilon^{ijk} [\lambda_i, \lambda_j] B_{ij} \Phi_k + \frac{i}{2} \epsilon_{ijk} [\bar{\lambda}^i, \bar{\lambda}^j] B_{ij} \Phi^k \right)$$

where $[\Phi_i, \Phi_j] C_{ij} = e^{iC_{ij}} \Phi_i \Phi_j - e^{-iC_{ij}} \Phi_j \Phi_i$ and $[\lambda_A, \lambda_B] B_{AB} = e^{iB_{AB}} \lambda_A \lambda_B - e^{-iB_{AB}} \lambda_B \lambda_A$. Here $1 \leq i, j \leq 3$ and $1 \leq A, B \leq 4$. 


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SU(2|3)

- SU(2|3) sector, a subset of states of the PSU(2, 2|4) theory.
- The field content is: \( \Phi_J = \{ \phi_1, \phi_2, \phi_3; \psi_1, \psi_2 \} \).
  - \( |\phi_a\rangle = c^\dagger_a c^\dagger_4 |0\rangle \)
  - \( |\psi_\alpha\rangle = a^\dagger_\alpha c^\dagger_4 |0\rangle \)
- The generators of the SU(2|3) superalgebra are
  \[
  R^a_b = c^\dagger_b c^a - \frac{1}{3} \delta^a_b c^\dagger c^c, \quad L^{\alpha \beta} = a^\dagger_\beta a^\alpha - \frac{1}{2} \delta^{\alpha \beta} a^\dagger a^\gamma a^\gamma, \\
  D = c^\dagger c + \frac{3}{2} a^\dagger a^\gamma, \quad S^\gamma_c = c^\dagger a^\gamma, \quad Q^c_\gamma = a^\dagger c^c.
  \]
- Maximal subalgebra
  - Large enough for interesting structural features to arise.
  - Higher order length fluctuations.
Yangian Algebra $Y(SU(2|3))$: $J^A, Q^A, \ldots$

Defining relations

$$
\begin{align*}
\{ J^A, J^B \} &= f^{AB}{}_C J^C, \\
\{ J^A, Q^B \} &= f^{AB}{}_C Q^C, \\
\{ Q^A, \{ Q^B, J^C \} \} &= \alpha f^{AG}{}_D f^{BH}{}_E f^{CK}{}_F f_{GHK} J^{D} J^{E} J^{F}
\end{align*}
$$

The $J^A$ are SU(2|3) generators. The tree-level, first nonlocal Yangian generator is

$$
Q^A_0 = -f^A{}_{CB} \sum_{i<j} J^B_0(i) J^C_0(j).
$$
Standard Coproducts

A coproduct is a holomorphic map \( \Delta : \mathcal{A} \rightarrow \mathcal{A} \). Introduces the idea of single site, double site, etc. . . representations for the algebra \( \mathcal{A} \). The coproduct for the ordinary SU(2|3) generators

\[
\Delta J^A = J^A \otimes 1 + 1 \otimes J^A.
\]

and the coproduct to create the two-site Yangian generators

\[
\Delta Q^A = Q^A \otimes 1 + 1 \otimes Q^A - f^A_{\;CB} J^B \otimes J^C
\]

will be used to construct tree level representations.
Hamiltonian

The two-site Hamiltonian in terms of oscillators is

\[ H(1, 2) = \left( c_{a}^{\dagger}(1) c_{b}(2) - c_{b}(1) c_{a}^{\dagger}(2) \right) c^{b}(2) c^{a}(1) \]
\[ + \left( c_{a}(1) a_{\alpha}(2) + a_{\alpha}(1) c_{a}(2) \right) a^{\alpha}(2) c^{a}(1) \]
\[ + \left( a_{\alpha}(1) c_{a}(2) + c_{a}(1) a_{\alpha}(2) \right) c^{a}(2) a^{\alpha}(1) \]
\[ + \left( a_{\alpha}(1) a_{\beta}(2) + a_{\beta}(1) a_{\alpha}(2) \right) a^{\beta}(2) a^{\alpha}(1). \]

The commutation relations of the oscillators are \( \{c^{a}(i), c_{b}^{\dagger}(j)\} = \delta_{b}^{a} \delta_{ij} \)
and \([a^{\alpha}(i), a_{\beta}^{\dagger}(j)] = \delta_{\beta}^{\alpha} \delta_{ij}.\)
Quadratic Casimir

The quadratic Casimir of the subalgebra SU(2|3) is

\[ g_{AB} J^A J^B = \frac{1}{3} D^2 + \frac{1}{2} L^{\alpha \beta} L_{\alpha \beta} - \frac{1}{2} R^a b R^b a - \frac{1}{2} [Q^c, S^{\gamma c}] . \]

A Casimir of any algebra has the property

\[ [g_{AB} J^A J^B, J^C] = 0 \]

When acting on any two-particle state \( |\Phi_I \Phi_J\rangle \) the quadratic Casimir and the two-site Hamiltonian are equivalent,

\[ H(1, 2) |\Phi_I \Phi_J\rangle = g_{AB} J^A J^B |\Phi_I \Phi_J\rangle . \]
The two-particle eigenstates of the Hamiltonian form two towers, 13 symmetric \((H_{12}|\Phi_1 \Phi_2\rangle_+ = 0 \cdot |\Phi_1 \Phi_2\rangle_+)\) and 12 antisymmetric \((H_{12}|\Phi_1 \Phi_2\rangle_- = 2|\Phi_1 \Phi_2\rangle_-)\) eigenstates.

\[
|ab\rangle_\pm = -\left( c_a(1)c_b(2) \mp c_b(1)c_a(2) \right) c_4(1)c_4(2)|0\rangle, \\
|a\beta\rangle_\pm = \left( c_a(1)a_\beta(2) \mp a_\beta(1)c_a(2) \right) c_4(1)c_4(2)|0\rangle, \\
|\alpha\beta\rangle_\pm = \left( a_\alpha(1)a_\beta(2) \mp a_\beta(1)a_\alpha(2) \right) c_4(1)c_4(2)|0\rangle.
\]
Analysis of Yangian Symmetry

A property of the dilatation generator is \([D, J^A] = (\text{dim} J^A) J^A\). Also, \([D, Q^A] = (\text{dim} J^A) Q^A\). Expanding both the dilatation generator and the Yangian,

\[
[D, Q^A] = [D_0 + g_{YM}^2 D_2 + \cdots, Q_0^A + g_{YM} Q_1^A + g_{YM}^2 Q_2^A + \cdots].
\]

Group in powers of the Yang-Mills coupling to \(O(g^2)\)

\[
(\text{dim} J^A)(Q_0^A + g_{YM} Q_1^A + g_{YM}^2 Q_2^A) + g_{YM}^2 [D_2, Q_0^A] \approx (\text{dim} J^A) Q^A.
\]

That means \([D_2, Q_0^A]\) must give zero (or approximately). In the large \(N\) (planar limit) \(D_2\) is our spin chain Hamiltonian, \(H\).
Edge Effects

In PSU(2, 2|4), an explicit check of the commutator gives the lattice derivative or ‘edge effects’ of the system,

\[ [D_2, Q^A_0] = q^A ∼ 0, \]
\[ q^A_{1L} = J^A(1) - J^A(L). \]

Introduce the identity,

\[ Q^A_{12} = \frac{1}{4} \left[ g_{BC} J^B(1) J^C(2), q^A_{12} \right], \]

Recall, the quadratic Casimir is equivalent to the Hamiltonian when acting on states. For the two-site case,

\[ \left[ H(1, 2), Q^A_{12} \right] |Φ_IΦ_J⟩ = \frac{1}{4} \left[ H(1, 2)^2 q^A_{12} + q^A_{12} H(1, 2)^2 - 2H(1, 2) q^A_{12} H(1, 2) \right] |Φ_IΦ_J⟩. \]

\[ = q^A_{12} |Φ_IΦ_J⟩. \]
Yangian on Two-Particle States

Key

\[ \begin{align*}
J^A \\
Q^A \\
u(1) \\
su(2) \\
su(3) \\
supercharges
\end{align*} \]

\[ \begin{align*}
H(1,2) &= 0 \\
H(1,2) &= 2
\end{align*} \]
Deformed Hamiltonian

The deformed R matrix, a solution to the Yang-Baxter equation is

$$\tilde{R}^{KL}_{IJ}(u) = \frac{1}{u + i} \left( u e^{-iB_{IJ}} \mathcal{I}^{KL}_{IJ} + i \mathcal{P}^{KL}_{IJ} \right).$$

The identity and projection operators are $\mathcal{I}^{KL}_{IJ} = \delta^K_I \delta^L_J$ and $\mathcal{P}^{KL}_{IJ} = \delta_I^L \delta^K_J$. The deformed monodromy matrix is

$$\tilde{T}^J_{I; \beta_1 \ldots \beta_L} = \tilde{R}^{b_L-1 \beta_L}_{l \alpha_L} \tilde{R}^{b_L-2 \beta_{L-1}}_{b_{L-1} \alpha_{L-1}} \ldots \tilde{R}^{b_1 \beta_2}_{b_2 \alpha_2} \tilde{R}^{J \beta_1}_{b_1 \alpha_1} \exp \left[ i \pi \sum_{i=1}^{L} \sum_{j=1}^{i-1} ([\alpha_i] + [\beta_i]) [\alpha_j] \right]$$

The deformed transfer matrix is $\tilde{T}(u) = (-)^J [\cdot]^J \tilde{T}^J_{J}(u)$. The deformed Hamiltonian is derived as

$$\tilde{H} = -i \left( \tilde{T}(u) \right)^{-1} \frac{d}{du} \tilde{T}(u) \bigg|_{u=0}$$
Deformed Two-Site Hamiltonian

The two-site transfer matrix is

\[ \tilde{T}(u) = \tilde{R}_{b_1 \alpha_2}^{b_1 \beta_2} \tilde{R}_{b_1 \alpha_1}^{a_\beta_1} \exp \left[ i \pi \left( [\alpha_2] + [\beta_2] \right) [\alpha_1] \right]. \]

Using the prescribed method, we derive the deformed Hamiltonian to be

\[ \tilde{H} = \left( \delta_{\alpha_1}^{\beta_1} \delta_{\alpha_2}^{\beta_2} - \delta_{\alpha_2}^{\beta_1} \delta_{\alpha_1}^{\beta_2} e^{-iB_{\alpha_1 \alpha_2}} \right) + \left( \delta_{\alpha_1}^{\beta_1} \delta_{\alpha_2}^{\beta_2} - \delta_{\alpha_2}^{\beta_2} \delta_{\alpha_1}^{\beta_1} e^{-iB_{\alpha_2 \alpha_1}} \right) \]

\[ = \left( \tilde{H}_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} \right) + \left( \tilde{H}_{\alpha_2 \alpha_1}^{\beta_2 \beta_1} \right). \]

Define the deformed two-site Hamiltonians, \( \tilde{H}(1, 2) \equiv \tilde{H}_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} \) and \( \tilde{H}(2, 1) \equiv \tilde{H}_{\alpha_2 \alpha_1}^{\beta_2 \beta_1}. \)
Deformed Hamiltonian

The oscillator representation of the deformed Hamiltonian

\[
\tilde{H}_{12} = \left( c_a^\dagger(1)c_b^\dagger(2) - e^{-iB_{ab}}c_b^\dagger(1)c_a^\dagger(2) \right) c^b(2)c^a(1) \\
+ \left( c_a^\dagger(1)a_{\alpha}(2) + e^{-iB_{a\alpha}}a_{\alpha}(1)c_a^\dagger(2) \right) a^\alpha(2)c^a(1) \\
+ \left( a_{\alpha}(1)c_a^\dagger(2) + e^{-iB_{a\alpha}}c_a^\dagger(1)a_{\alpha}(2) \right) c^a(2)a^\alpha(1) \\
+ \left( a_{\alpha}(1)a_{\beta}^\dagger(2) + e^{-iB_{\alpha\beta}}a_{\beta}(1)a_{\alpha}^\dagger(2) \right) a^\beta(2)a^\alpha(1).
\]

\(B_{IJ}\) is a real, antisymmetric matrix.
Due to twisting, the Hamiltonian is no longer a Casimir of the SU(2|3) superalgebra

$$\left[ \tilde{H}(1, 2), J^A_{12} \right] \neq 0,$$

nor do we generally have the edge effects

$$\left[ \tilde{H}(1, 2), Q^A_{12} \right] \neq q^A_{12},$$

when using the previous (coproduct) construction for $J^A_{12}$ and $Q^A_{12}$. 
The Reshitikhin twist is generated by a deforming function $F$. The R-matrix deforms as

$$\tilde{R}(u) = FR(u)F^{-1}.$$ 

The coproduct receives a deformation as well,

$$\Delta^{(F)} = F\Delta F^{-1}.$$ 

The deforming function has the definition

$$F = \exp\left[\frac{i}{2} \sum_{I<J} B_{IJ} \left( E^{II} \otimes E^{JJ} - E^{JJ} \otimes E^{II} \right) \right]$$
Twisted Coproducts

Twisted coproducts $\Delta J^A_B = K_{AB} \otimes J^A_B + J^A_B \otimes K_{BA}$.

$$
\begin{align*}
\Delta R^a_b &= K_{ab} \otimes R^a_b + R^a_b \otimes K_{ba}, \\
\Delta L^\alpha{}^\beta &= K^\alpha{}^\beta \otimes L^\alpha{}^\beta + L^\alpha{}^\beta \otimes K^\beta{}^\alpha, \\
\Delta Q^c{}^\gamma &= K^c{}^\gamma \otimes Q^c{}^\gamma + Q^c{}^\gamma \otimes K^\gamma{}^c, \\
\Delta S^\gamma{}^c &= K^\gamma{}^c \otimes S^\gamma{}^c + S^\gamma{}^c \otimes K^c{}^\gamma, \\
\Delta D &= 1 \otimes D + D \otimes 1.
\end{align*}
$$

The twisting is brought about by

$$
K_{IJ} = \exp \left[ \frac{i}{2} \sum_{K=1}^{5} (B_{IK} - B_{JK}) E_{KK} \right].
$$

We define matrices $(E_{IJ})_{KL} = \delta_{IK} \delta_{JL}$ which satisfy $[E_{IJ}, E_{KL}] = \delta_{KJ} E_{IL} - \delta_{IL} E_{AJ}$. The $E_{IJ}$ are the generators of $U(2|3)$. 
Yangians Via Twisted Coproducts

The first Yangian construction via coproducts is

\[ \Delta Q^I_J \sim K_{IJ} \otimes Q^I_J + Q^I_J \otimes K_{JI} \]

\[ + \frac{1}{2} \sum_{K=1}^{5} \left( J^I_K K_{KJ} \otimes K_{KI} J^K_J - K_{IK} J^K_J \otimes J^I_K K_{JK} \right) \]

An example two-site Yangian is

\[ \Delta Q^{(R^a_b)} a b = K_{ab} \otimes Q^{(R^a_b)} a b + Q^{(R^a_b)} a b \otimes K_{ba} \]

\[ + \frac{1}{2} \left( R^a_c K_{cb} \otimes K_{ca} R^c_b - K_{ac} R^c_b \otimes R^a_c K_{bc} \right) \]

\[ + \frac{1}{2} \left( Q^a_\gamma K_{\gamma b} \otimes K_{\gamma a} S^\gamma b + K_{a_\gamma} S^\gamma b \otimes Q^a_\gamma K_{b_\gamma} \right) \]

\[ - \frac{1}{6} \delta^a_b \left( Q^c_\gamma K_{\gamma c} \otimes K_{\gamma c} S^\gamma c + K_{c_\gamma} S^\gamma c \otimes Q^c_\gamma K_{c_\gamma} \right) \]
It Works

Acting on two particle states, the deformed Hamiltonian is equivalent to the (deformed) Casimir

\[ \Delta J^A_B \Delta J^B_A |\Phi_I \Phi_J\rangle = \tilde{H}(1, 2) |\Phi_I \Phi_J\rangle. \]

We check the one-loop calculation of the dilatation generator again,

\[ \left[ \tilde{H}(1, 2), Q^A_{12B} \right] = \tilde{q}^A_{12B}, \]

where the edge effect term, \( \tilde{q}^A_{12B} \), has a deformation dependence

\[ \tilde{q}^A_{12B} = J^A_B \otimes K_{AB} - K_{BA} \otimes J^A_B. \]

We find for an infinite length spin chain when \( J^A_B(1), J^A_B(L) \rightarrow 0 \), then \( \tilde{q}^A_{12B} \rightarrow 0 \). Recall, this is what we want!
Residual SU(2) × U(1)^3 symmetry. This is the beta deformation of Lunin-Maldacena. \( B_{13} = B_{21} = B_{32} = \gamma. \)

**Residual Symmetry**

\[
\Delta L^{\alpha\beta} = 1 \otimes L^{\alpha\beta} + L^{\alpha\beta} \otimes 1,
\]
\[
\Delta D = 1 \otimes D + D \otimes 1,
\]
\[
\Delta R^c_c = 1 \otimes R^c_c + R^c_c \otimes 1,
\]

**Remaining Symmetry**

\[
\Delta R^a_b = K_{ab} \otimes R^a_b + R^a_b \otimes K_{ba},
\]
\[
\Delta Q^c_\gamma = K_{c\gamma} \otimes Q^c_\gamma + Q^c_\gamma \otimes K_{\gamma c},
\]
\[
\Delta S^\gamma_c = K_{\gamma c} \otimes S^\gamma_c + S^\gamma_c \otimes K_{c\gamma}.\]
Deformations Maintaining $\mathcal{N} = 1$ SCFT: Case 2

Residual SU(2|1) $\times$ U(1)$^2$ symmetry.

$B_{12} = B_{13} = B_{23} = B_{1\alpha} = -B_{2\alpha} = \gamma$.

Residual Symmetry

$$
\Delta L^\alpha_\beta = 1 \otimes L^\alpha_\beta + L^\alpha_\beta \otimes 1,
\Delta Q^3_\gamma = 1 \otimes Q^3_\gamma + Q^3_\gamma \otimes 1,
\Delta S^{\gamma}_3 = 1 \otimes S^{\gamma}_3 + S^{\gamma}_3 \otimes 1,
\Delta D = 1 \otimes D + D \otimes 1,
\Delta R^c_c = 1 \otimes R^c_c + R^c_c \otimes 1,
$$

Remaining Symmetry

$$
\Delta R^a_b = K_{ab} \otimes R^a_b + R^a_b \otimes K_{ba},
\Delta Q^c_\gamma = K_{c\gamma} \otimes Q^c_\gamma + Q^c_\gamma \otimes K_{c\gamma},
\Delta S^{\gamma}_c = K_{\gamma c} \otimes S^{\gamma}_c + S^{\gamma}_c \otimes K_{c\gamma}.
$$
Conclusion

- Constructed a Yangian algebra for SU(2|3)
- The deformed theory corresponds to a deformed R-matrix
- Introduced twisted coproducts corresponding to the twisted R-matrix
- Showed that twisted coproducts could be used to describe broken symmetry in $\mathcal{L}$
- The deformed Hamiltonian (one-loop dilatation generator) still has SU(2|3) Yangian symmetry but with twisted coproduct
\[ \Delta Q_{(R^a)}^{\{R^a\}}_{ab} = K_{ab} \otimes Q_{(R^a)}^{\{R^a\}}_{ab} + Q_{(R^a)}^{\{R^a\}}_{ab} \otimes K_{ba} + \frac{1}{2} h (R^a_c K_{cb} \otimes K_{ca} R^c_b - K_{ac} R^c_b \otimes R^a_c K_{bc}) + \frac{1}{2} h (Q^a_\gamma K_{\gamma b} \otimes K_{\gamma a} S^\gamma_b + K_{a\gamma} S^\gamma_b \otimes Q^a_\gamma K_{b\gamma}) - \frac{1}{6} h \delta^a_b (Q^c_\gamma K_{\gamma c} \otimes K_{\gamma c} S^\gamma_c + K_{c\gamma} S^\gamma_c \otimes Q^c_\gamma K_{c\gamma}), \]

\[ \Delta Q_{(L^\alpha)}^{\{L^\alpha\}}_{\alpha \beta} = K_{\alpha \beta} \otimes Q_{(L^\alpha)}^{\{L^\alpha\}}_{\alpha \beta} + Q_{(L^\alpha)}^{\{L^\alpha\}}_{\alpha \beta} \otimes K_{\beta \alpha} + \frac{1}{2} h (L^\alpha_\gamma K_{\gamma \beta} \otimes K_{\gamma \alpha} L^\gamma_\beta - K_{\alpha \gamma} L^\gamma_\beta \otimes L^\alpha_\gamma K_{\beta \gamma}) + \frac{1}{2} h (S^\alpha_{c\beta} \otimes K_{c\alpha} Q^c_\beta + K_{\alpha c} Q^c_\beta \otimes S^\alpha_{c} K_{\beta c}) - \frac{1}{4} h \delta^\alpha_\beta (S^\gamma_c K_{c\gamma} \otimes K_{c\gamma} Q^c_\gamma + K_{\gamma c} Q^c_\gamma \otimes S^\gamma_c K_{c\gamma}), \]
\[
\Delta Q_{(Q^a_\alpha)}^{a_\alpha} = K_{a\alpha} \otimes Q_{(Q^a_\alpha)}^{a_\alpha} + Q_{(Q^a_\alpha)}^{a_\alpha} \otimes K_{a\alpha} \\
+ \frac{1}{2} h \left( Q^a_\gamma K_{\gamma\alpha} \otimes K_{\gamma\alpha} L^{\gamma\alpha} - K_{\alpha\gamma} L^{\gamma\alpha} \otimes Q^a_\gamma K_{\alpha\gamma} \right) \\
+ \frac{1}{2} h \left( R^a_c K_{ca} \otimes K_{ca} Q^c_\alpha - K_{ac} Q^c_\alpha \otimes R^a_c K_{ac} \right),
\]

\[
\Delta Q_{(S^\alpha_a)}^{\alpha a} = K_{\alpha a} \otimes Q_{(S^\alpha_a)}^{\alpha a} + Q_{(S^\alpha_a)}^{\alpha a} \otimes K_{\alpha a} \\
+ \frac{1}{2} h \left( S^\alpha_c K_{ca} \otimes K_{ca} R^c_{\alpha} - K_{\alpha c} R^c_{\alpha} \otimes S^\alpha_c K_{ac} \right) \\
+ \frac{1}{2} h \left( L^\alpha_\gamma K_{\gamma a} \otimes K_{\gamma a} S^{\gamma a} - K_{\alpha\gamma} S^{\gamma a} \otimes L^\alpha_\gamma K_{\alpha a} \right),
\]

\[
\Delta Q_{(D)} = 1 \otimes Q_{(D)} + Q_{(D)} \otimes 1 \\
+ \frac{1}{4} h \left( S^{\gamma c} K_{c\gamma} \otimes K_{c\gamma} Q^c_{\gamma} + K_{\gamma c} Q^c_{\gamma} \otimes S^{\gamma c} K_{c\gamma} \right).
\]