On the duality between CS-matter theory and strings in $\text{AdS}_4 \times \text{CP}^3$: loops vs. spins

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based on work with T. McLoughlin and A. Tseytlin
Many reasons to study 3d CFT-s:

- potential revelations on the M2-brane theory
  attempts by Bagger, Lambert

- fixed points of condensed matter systems

- understanding of part of the landscape of $d = 4$ string vacua

- potentially tractable examples of gauge/string duality
On the M2-brane theory

- AdS/CFT: theory is conformal and dual to M-theory on $AdS_4 \times S^7$
  - fixed point of the D2 brane theory
    - 8 physical scalars
    - perhaps additional, topological degrees of freedom
  - 3d gauge theory has dimensionful coupling $\rightarrow$ must disappear at the fixed point $\rightarrow$ only CS-type quadratic term

- Parameters: 't Hooft coupling: $\lambda = g_{YM}^2 N \rightarrow \lambda_{\text{CS}} = \frac{N}{k_{\text{CS}}}$

- Interpretation of level $k_{\text{CS}}$? Natural values? 10d connection?
Outline

- The $\mathcal{N} = 6$ CS-matter theory
- The conjectured Bethe ansatz and its relation to $\text{AdS}_5 \times S^5$
- Worldsheets calculations, comparison and differences
- Outlook
$U(N) \times U(N)$ Chern-Simons-matter theory with $\mathcal{N} = 6$ susy
– special case of $\mathcal{N} = 3$ construction

- $SO(6) \simeq SU(4)$ R-symmetry
- 4 complex scalar fields: $Y^A \in \mathbb{N} \times \bar{\mathbb{N}}$ and $Y^\dagger_A \in \bar{\mathbb{N}} \times \mathbb{N}$
- 4 complex fermions
- supermultiplet: scalars in 4 and fermions in $\bar{4} \mapsto$ susy gen’s in 6

\[
S = \frac{k_{cs}}{4\pi} \int d^3 x \text{Tr} \left[ \epsilon^{\mu\nu\rho} (A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho - \hat{A}_\mu \partial_\nu \hat{A}_\rho - \frac{2}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\rho) \\
+ D_\mu Y^\dagger_A D^\mu Y^A + \frac{1}{12} Y^A Y^\dagger_A Y^B Y^\dagger_B Y^C Y^\dagger_C + \frac{1}{12} Y^A Y^\dagger_B Y^B Y^\dagger_C Y^C Y^\dagger_A \\
- \frac{1}{2} Y^A Y^\dagger_A Y^B Y^\dagger_C Y^C Y^\dagger_B + \frac{1}{3} Y^A Y^\dagger_B Y^C Y^\dagger_A Y^B Y^\dagger_C + \text{fermions} \right]
\]

- superpotential $W = \epsilon^{ab} \epsilon^{\dot{a}\dot{b}} \text{Tr} [A_a B_{\dot{a}} A_{\dot{b}} B_{\dot{b}}]$; $Y^A = (A_1, A_2, B^\dagger_1, B^\dagger_2)$
- Covariant derivative: $D_\mu Y^A = \partial_\mu Y^A + A_\mu Y^A - Y^A \hat{A}_\mu$
\[ S = \frac{k_{CS}}{4\pi} \int d^3x \text{Tr} \left[ \epsilon^{\mu\nu\rho} (A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho - \tilde{A}_\mu \partial_\nu \tilde{A}_\rho - \frac{2}{3} \tilde{A}_\mu \tilde{A}_\nu \tilde{A}_\rho) 
right.
\]
\[ + D_\mu Y_A^\dagger D^\mu Y_A + \frac{1}{12} Y^A Y_B^\dagger Y_C^\dagger Y_C^\dagger + \frac{1}{12} Y^A Y_B^\dagger Y_B^\dagger Y_C^\dagger Y_C^\dagger \]
\[ - \frac{1}{2} Y^A Y_B^\dagger Y_C^\dagger Y_B^\dagger + \frac{1}{3} Y^A Y_B^\dagger Y_C^\dagger Y_A^\dagger Y_B^\dagger + \text{fermions} \]

- Power-counting renormalizable; special choice of levels \( k_1 = -k_2 \)

- **Planar perturbation theory:** Series expansion in \( \lambda^2 \) rather than \( \lambda \) (a feature of 3d perturbation theory)

- Argued to have exact conformal invariance – \( OSp(6|4) \) symmetry

- Theory constructible from \( N = 4 \ d = 2 + 1 \) SYM theory broken to \( N = 3 \) and deformed by supersymmetric CS term and flown to \( E \ll m = g_{YM}^2 k_{CS}/(4\pi) \)  

  Aharony, Bergman, Jafferis, Maldacena
String/M-theory dual: almost-max susy, correct symmetries

- $AdS_4 \times \mathbb{CP}^3$ has $SO(3, 2) \times SO(6) \simeq Sp(4) \times SO(6)$ symmetry
- $\mathbb{Z}_k$ orbifold projection of $AdS_4 \times S^7$ on nonsingular fiber
  \[ S^1 \rightarrow S^7 \]
  \[ \mathbb{CP}^3 \]
- M2-branes on $\mathbb{C}^4 / \mathbb{Z}_k$ (weak coupling stability ensured by supersymmetry)
- string theory limit: $k \rightarrow \infty$ relate $k$ and $k_{cs}$

\[
\begin{align*}
    ds^2_{AdS_4 \times S^7} &= \frac{R^4}{4} \left( ds^2_{AdS_4} + 4 ds^2_{S^7} \right) \quad F_{(4)} \propto \text{Vol}(AdS_4) \\
    ds^2_{S^7} &= (d\phi + \omega)^2 + ds^2_{\mathbb{CP}^3} \quad \mathbb{Z}_k \quad ds^2 = \frac{1}{k^2} (d\phi + k\omega)^2 + ds^2_{\mathbb{CP}^3}
\end{align*}
\]

- Account for volume reduction:

\[
\begin{align*}
    ds^2 &= \frac{R^3}{4k_{cs}} \left( ds^2_{AdS_4} + 4 ds^2_{\mathbb{CP}^3} \right) \quad e^{2\phi} = \frac{R^3}{k_{cs}^3} \\
    F_2 &= k_{cs} \mathbb{J}_{\mathbb{CP}^3} \quad F_4 = \frac{3}{8} R^3 \text{Vol}_{AdS_4}
\end{align*}
\]
So here is another conjectured gauge/string duality. Why bother?

- **less-than maximal susy**: may exhibit features absent in $\text{AdS}_5 \times S^5$
  
  - different coupling constant dependence
  
  - fewer protected quantities; **more interpolating functions**

- Tractable both at weak and strong coupling and thus testable

Where to begin?

- expect agreement for all quantities protected by symmetries

  $\rightarrow$ focus on unprotected quantities – *e.g.* anomalous dimensions
Leading order dilatation operator for scalar operators Minahan, Zarembo

- main difference from $\mathcal{N} = 4$ SYM: scalars in bifundamental rep.
  $\mapsto$ gauge-invariant scalar operators are of the type
  \[
  \text{Tr} \left[ Y^{A_1}_{B_1} Y^{A_2}_{B_2} Y^{A_3}_{B_3} \ldots Y^{A_L}_{B_L} \right]
  \]
- arises at 2-loops
  - has nearest and next-to-nearest neighbor interactions
  \[
  \Gamma = \frac{\lambda^2}{2} \sum_{l=1}^{2L} H_{l,l+1,l+2}
  \]
  \[
  H_{l,l+1,l+2} = 1 - K_{l,l+1} - 2P_{l,l+2} + P_{l,l+2}K_{l,l+1} + K_{l,l+1}P_{l,l+2}
  \]
- Trace and permutation operators:
  \[
  K : V \times \bar{V} \rightarrow V \times \bar{V} \quad K_{BA'}^{AB} = \delta_{AB'}\delta^{BA'}
  \]
  \[
  P : V \times V \rightarrow V \times V \quad P_{A'B'}^{AB} = \delta_{B'}^{A'}\delta^{B}_{A'}
  \]
- and the surprise is...
... that, despite the next-to-nearest neighbor interaction, this operator may be identified with a Hamiltonian derived from monodromy matrices obeying the Yang-Baxter equation and thus is integrable

– one for even sites: \[ T_a(u, \alpha) \propto R_{aq_{1}}(u)R_{a\bar{q}_{1}}(u + \alpha) \ldots R_{aq_{L}}(u)R_{a\bar{q}_{L}}(u + \alpha) \]

– one for odd sites: \[ \bar{T}_{a}(u, \alpha) \propto R_{\bar{a}q_{1}}(u + \alpha)R_{\bar{a}\bar{q}_{1}}(u) \ldots R_{\bar{a}q_{L}}(u + \alpha)R_{\bar{a}\bar{q}_{L}}(u) \]

- 1-loop dilatation operator is recovered by choosing \( \alpha = -2 \)

\[ \tau = \text{Tr} \left[ T_{a} \right] \quad \bar{\tau} = \text{Tr} \left[ \bar{T}_{a} \right] \quad [\tau, \bar{\tau}] = 0 \quad H_{\text{even}} = \tau^{-1}d_u \tau \quad H_{\text{odd}} = \bar{\tau}^{-1}d_u \bar{\tau} \]

Assuming all-order integrability: use machinery of discrete integrable models and symmetries preserved by the lowest dimension operator

– understand closed sectors (subsets of operators closed under RG flow)

– construct spin chain S-matrix (solve Yang-Baxter equation)

– construct Bethe ansatz \( \rightarrow \) Bethe equations

– understand coupling constant dependence
Closed sectors – should be determined by symmetries

– difference from $\mathcal{N} = 4$ SYM: both scalars and fermions are in the same representation of the R-symmetry group!

$\mapsto$ 2 scalars and 1 derivative $\Leftrightarrow$ 2 fermions

$\mapsto$ departure from familiar closed sectors
Closed sectors – should be determined by symmetries

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  in the same representation of the R-symmetry group!

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$\mapsto$ departure from familiar closed sectors

S-matrix: vacuum $\text{Tr} \ [(Y^1 Y^\dagger_4)^L]$ preserves $SU(2|2) \subset OSp(6|4)$

- alternating chain $\rightarrow$ separate excitations on even and odd sites
- rep’s of $SU(2|2)$; conjectured to be $(2|2)$
  
  $Y^1 \rightarrow (Y^2, Y^3|(\psi_3)_\alpha)$ (A-ext’s) and $Y^\dagger_4 \rightarrow (Y^\dagger_2, Y^\dagger_3|(\psi^\dagger_2)_\alpha)$ (B-ext’s)

- 3 S-matrices: $S_{AA}$, $S_{BB}$ and $S_{AB}$

Beisert’s $psu(2|2)$ S-matrix less clear

Excitation energy: $\epsilon(p) = \sqrt{\frac{1}{4} + 4\pi^2 \hbar^2(\lambda) \sin^2 \frac{p}{2}}$ for both
Closed sectors – should be determined by symmetries

– difference from $\text{AdS}_5 \times S^5$: both scalars and fermions are in the same representation of the R-symmetry group!

$\leftrightarrow$ 2 scalars and 1 derivative $\leftrightarrow$ 2 fermions

$\leftrightarrow$ departure from $\text{AdS}_5 \times S^5$ sectors

S-matrix: vacuum $\text{Tr} \ [(Y^1 Y^\dagger_4)^L]$ preserves $SU(2|2) \subset OSP(6|4)$

- alternating chain $\rightarrow$ separate excitations on even and odd sites
- rep’s of $SU(2|2)$; conjectured to be $(2|2)$

$Y^1 \rightarrow (Y^2, Y^3|\psi_3^\alpha) \ (\text{A-ext's})$ and $Y^\dagger_4 \rightarrow (Y^\dagger_2, Y^\dagger_3|\psi_2^\dagger_\alpha) \ (\text{B-ext's})$

- 3 S-matrices: $S_{AA}$, $S_{BB}$ and $S_{AB}$

- Formal similarity w/ S-matrices of CFT-s (e.g. Z’s S-matrix for WZW) if one identifies $A$ and $B$ excitations with left- and right-movers.

- $(2|2) \oplus (2|2)$ excitations $\rightarrow$ formal difference with expected number of excitations on the worldsheet where there are $(8|8)$ physical fields
The Bethe equations

\[
1 = \prod_{j=1}^{K_2} \frac{u_{1,k} - u_{2,j} + \frac{i}{2}}{u_{1,k} - u_{2,j} - \frac{i}{2}} \prod_{j=1}^{K_1} \frac{1 - 1/x_{1,k} x_{4,j}^+}{1 - 1/x_{1,k} x_{4,j}^-} \prod_{j=1}^{K_4} \frac{1 - 1/x_{1,k} x_{4,j}^+}{1 - 1/x_{1,k} x_{4,j}^-},
\]

\[
1 = \prod_{j \neq k}^{K_2} \frac{u_{2,k} - u_{2,j} - i}{u_{2,k} - u_{2,j} + i} \prod_{j=1}^{K_1} \frac{u_{2,k} - u_{1,j} + \frac{i}{2}}{u_{2,k} - u_{1,j} - \frac{i}{2}} \prod_{j=1}^{K_3} \frac{u_{1,k} - u_{3,j} + \frac{i}{2}}{u_{1,k} - u_{3,j} - \frac{i}{2}},
\]

\[
1 = \prod_{j=1}^{K_2} \frac{u_{3,k} - u_{2,j} + \frac{i}{2}}{u_{3,k} - u_{2,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{3,k} - x_{4,j}^+}{x_{3,k} - x_{4,j}^-} \prod_{j=1}^{K_1} \frac{x_{3,k} - x_{4,j}^+}{x_{3,k} - x_{4,j}^-} \times \prod_{j=1}^{K_4} \sigma_{\text{BES}}(u_{4,k}, u_{4,j}) \prod_{j=1}^{K_4} \sigma_{\text{BES}}(u_{3,k}, u_{3,j}),
\]

\[
\left( \frac{x_{4,k}^+}{x_{4,k}^-} \right)^L = \prod_{j \neq k}^{K_4} \frac{u_{4,k} - u_{4,j} + i}{u_{4,k} - u_{4,j} - i} \prod_{j=1}^{K_1} \frac{1 - 1/x_{4,k} x_{3,j}^+}{1 - 1/x_{4,k} x_{3,j}^-} \prod_{j=1}^{K_3} \frac{x_{3,k} - x_{4,j}^-}{x_{3,k} - x_{4,j}^+} \times \prod_{j=1}^{K_4} \sigma_{\text{BES}}(u_{4,k}, u_{4,j}) \prod_{j=1}^{K_4} \sigma_{\text{BES}}(u_{3,k}, u_{3,j}).
\]

\[
E = \frac{1}{2} \sum_{j=1}^{K_4} \left( \sqrt{1 + 16h(\lambda)^2 \sin^2 \frac{p_j}{2}} - 1 \right) + \frac{1}{2} \sum_{j=1}^{K_3} \left( \sqrt{1 + 16h(\lambda)^2 \sin^2 \frac{\bar{p}_j}{2}} - 1 \right)
\]

\[
p_j = \frac{1}{i} \log \frac{x_{4,j}^+}{x_{4,j}^-}
\]

\[
\bar{p}_j = \frac{1}{i} \log \frac{x_{4,j}^+}{x_{4,j}^-}
\]
Apparently a truncation is possible: set $K_1, K_2, K_3 = 0; K_4 = \bar{K}_4$

\[
\left( \frac{x_k^+}{x_k^-} \right)^L = - \prod_{j \neq k}^S \frac{u_k - u_j + i \left( \frac{x_k^- - x_j^+}{x_k^+ - x_j^-} \right)^2 \sigma_{\text{BES}}^2(u_k, u_j)}{u_k - u_j - i \left( \frac{x_k^+ - x_j^-}{x_k^- - x_j^+} \right)^2} \quad 1 = \left( \prod_{j=1}^S \frac{x_j^+}{x_j^-} \right)^2
\]

• Energy: $E = \sum_{j=1}^S \sqrt{1 + 16h(\lambda)^2 \sin^2 \frac{p_j}{2}} \quad h(\lambda)^2 = \lambda^2 + \mathcal{O}(\lambda^4)$

- Suggested eq's for $SL(2)$ sector – spin $S$ and R-charge $L = 2J$

Gromov, Vieira

- many similarities with Bethe eq's for the $SL(2)$ sector of AdS$_5 \times$S$^5$

The map:
- $\sqrt{\lambda} \mapsto 4\pi h(\lambda)$
- Bethe mode number shifted by $1/2$
- $E_{\text{AdS}_5} \mapsto 2E_{\text{AdS}_4}$ (twice as many excitations)
- $S_{\text{AdS}_5} \mapsto 2S_{\text{AdS}_4}$ (BPS relation)
Bethe Ansatz vs. The Worldsheet

**eternal problem:** how to do reliable worldsheet perturbation theory and identify correctly the gauge theory and string theory parameters

**eternal solution:** Focus on states with large quantum numbers; worldsheet semiclassical expansion is reliable; identify the gauge theory operator by matching its charges; the charge and the “size” of the worldsheet are related
Bethe Ansatz vs. The Worldsheet

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◊ **Two important solutions:**

1) spinning folded string  
   GKP; Frolov, Tseytlin

2) circular rotating string with 2 angular momenta  
   Park, Tirziu, Tseytlin
   - both exist in $\text{AdS}_3 \times S^1 \subset \text{AdS}_5 \times S^5$ and $\text{AdS}_4 \times \mathbb{CP}^3$
   - both exhibit minimal structural changes compared to $\text{AdS}_5 \times S^5$
   - main difference related to RR fields
   - potentially expose subtle differences between the two models
The action: Bosonic part: sigma model based on the metrics

$$ds^2_{\text{AdS}_4} = - \cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \left( d\theta^2 + \sin^2 \theta d\phi^2 \right)$$

$$ds^2_{\text{CP}^3} = d\zeta_1^2 + \sin^2 \zeta_1 \left[ d\zeta_2^2 + \cos^2 \zeta_1 \left( d\tau_1 + \sin^2 \zeta_2 \left( d\tau_2 + \sin^2 \zeta_3 d\tau_3 \right) \right)^2 
+ \sin^2 \zeta_2 \left( d\zeta_3^2 + \cos^2 \zeta_2 \left( d\tau_2 + \sin^2 \zeta_3 d\tau_3 \right)^2 + \sin^2 \zeta_3 \cos^2 \zeta_3 d\tau_3^2 \right) \right]$$

- Coordinates iteratively embedding $\mathbb{C}P^{n-1}$ into $\mathbb{C}P^n$  

- Radii: $R_{\text{CP}^3}^2 = 4R_{\text{AdS}}^2$  
  $R_{\text{AdS}}^2 = \frac{R^3}{4k_{\text{CS}}} = \pi \sqrt{2\lambda} = \sqrt{\bar{\lambda}} \equiv$ string tension

Fermionic part: complete all-order GS action is not clear

V1. Use $\text{AdS}_4 \times \mathbb{C}P^3 = SO(3, 2)/SO(3, 1) \times SU(4)/SU(3) \times U(1)$ and fit in a supergroup: $OSp(6|4)/SO(3, 1) \times SU(4)/SU(3) \times U(1)$  

- only 24 fermions; partial $\kappa$-gauge-fixed; needs motion on $\mathbb{C}P^3$

V2. Double dimensional reduction from supermembrane in $\text{AdS}_4 \times S^7$

V3. Perturbative construction in number of fermions (need only $\theta^2$)
The action: Bosonic part: sigma model based on the metrics

\[ ds^2_{\text{AdS}_4} = -\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \]

\[ ds^2_{\mathbb{C}P^3} = d\zeta_1^2 + \sin^2 \zeta_1 \left[ d\zeta_2^2 + \cos^2 \zeta_1 \left( d\tau_1 + \sin^2 \zeta_2 \left( d\tau_2 + \sin^2 \zeta_3 \, d\tau_3 \right) \right)^2 \right. \]

\[ + \left. \sin^2 \zeta_2 \left( d\zeta_3^2 + \cos^2 \zeta_2 \left( d\tau_2 + \sin^2 \zeta_3 \, d\tau_3 \right)^2 + \sin^2 \zeta_3 \cos^2 \zeta_3 \, d\tau_3^2 \right) \right] \]

- Coordinates iteratively embedding \( \mathbb{C}P^{n-1} \) into \( \mathbb{C}P^n \) \( \text{Hoxha et al} \)
- Radii: \( R_{\mathbb{C}P^3}^2 = 4R_{\text{AdS}}^2 \) \( R_{\text{AdS}}^2 = \frac{R^3}{4k_{cs}} = \pi\sqrt{2}\lambda = \sqrt{\bar{\lambda}} \equiv \text{string tension} \)

Fermionic part: complete all-order GS action is not clear

\( \text{V1. GS on } OSp(6|4)/SO(3,1) \times SU(4)/SU(3) \times U(1) \) \( \text{Arutyunov, Frolov; Stefanski; Fre, Grassi} \)

- only 24 fermions; partial \( \kappa \)-gauge-fixed; needs motion on \( \mathbb{C}P^3 \)
- Clasically integrable; classical transfer matrix
- Interesting open quantum question: conservation of higher charges is anomalous in sigma models on \( \mathbb{C}P^n \) and cancels in ws susy situations; are GS fermions equally powerful?

- Assume all is well; discretize classical BE; conjecture all-order \( \text{Gromov, Vieira} \)
Semiclassical expansion:

\[
S = \frac{R_{\text{AdS}}^2}{2\pi} \int d\tau \int_0^{2\pi} d\sigma \sqrt{-g} g^{ab} \frac{1}{2} \partial_a X^M \partial_a X^N G_{MN}(X) \quad R_{\text{AdS}}^2 = \sqrt{\bar{\lambda}}
\]

- \( \bar{\lambda} = \lambda \) in AdS\(_5\times S^5\) while \( \bar{\lambda} = 2\pi^2 \lambda \) in AdS\(_4\times\mathbb{CP}^3\)

Target space energy density

\[
E = \sqrt{\bar{\lambda}} \mathcal{E} \left( S_i, J_i, \frac{1}{\sqrt{\bar{\lambda}}} \right) = \sqrt{\bar{\lambda}} \left[ \mathcal{E}_0 (S_i, J_i) + \frac{1}{\sqrt{\bar{\lambda}}} \mathcal{E}_1 (S_i, J_i) + \ldots \right]
\]

Spin density R-charge density \( S_i = \sqrt{\bar{\lambda}} S_i \quad J_i = \sqrt{\bar{\lambda}} J_i \)

- Charges = identify the Cartan-s; phases of embedding coord’s

Magnon dispersion relation at strong coupling: \( \exists 8 \) bosonic exc.

- BMN limit using one of the Cartan isometries

\[
\epsilon_{L,H} = \sqrt{n_{L,H} + 4\pi^2 h(\lambda)^2 k^2 / j^2} \quad h(\lambda) = \sqrt{\frac{\lambda}{2}} + \mathcal{O}(1)
\]

- Bethe ansatz: leading correction to \( h(\lambda) \) vanishes

Nishioka, Takayanagi

Shenderovich
- Tractable limit of the spinning folded string with finite charges: 
  $S \gg J \gg 1 \quad l = \frac{J}{\sqrt{\lambda} \ln S} = \text{fixed} \rightarrow \text{homogeneous in w.s. coordinates}$

  $\bar{t} = \kappa \tau \quad \bar{\rho} = \mu \sigma \quad \bar{\phi} = \kappa \tau \quad \bar{\phi}_2 = \bar{\phi}_3 = \frac{1}{2} \nu \tau \quad \mu^2 = \kappa^2 - \nu^2$

  $(\mathcal{E}, S, J) = \int_0^{2\pi} d\sigma \frac{1}{2}(\kappa \cosh^2 \bar{\rho}, \kappa \sinh^2 \bar{\rho}, \nu)$ \quad Virasoro constraints

  $\mu = \frac{1}{\pi} \ln S \quad \mu \gg 1 \quad l = \frac{\nu}{\mu}$ can define $\mu \sigma$ as spatial ws coordinate

  $\rightarrow$ string length is effectively infinite

  $\rightarrow$ $\mu$-dependence factorizes

- Leading order value of the space-time energy

  $E_0 - S = \sqrt{\lambda} \ln S \sqrt{1 + l^2} = \sqrt{\lambda} f_0(l) \ln S$

- General behavior: $E - S = \sqrt{\lambda} f(\bar{\lambda}, l) \ln S$ 

  $\uparrow$ universal scaling function
Circular rotating string:

\[ \bar{t} = \kappa \tau \quad \bar{\rho} = \rho_* \quad \bar{\theta} = \frac{\pi}{2} \quad \bar{\phi} = w \tau + k \sigma \quad \bar{\varphi}_2 = \bar{\varphi}_3 = \frac{1}{2}(\omega \tau + m \sigma) \]

- Virasoro constraints and eq’s of motion \((r_0 \equiv \cosh \rho_* \text{ and } r_1 \equiv \sinh \rho_*)\)

\[ w^2 - (\kappa^2 + k^2) = 0 \quad r_1^2 w k + \omega m = 0 \quad r_0^2 \kappa^2 - r_1^2 (w^2 + k^2) - \omega^2 - m^2 = 0 \]

- Classical energy and charges

\[ E_0 = \sqrt{\bar{\lambda}} r_0^2 \kappa \quad S = \sqrt{\bar{\lambda}} r_1^2 w \quad J \equiv J_2 = J_3 = \sqrt{\bar{\lambda}} \omega \]

- Express \(E_0\) in terms of charges and winding numbers \(k\) and \(m\) in the scaling limit \(S, J \rightarrow \infty\) with \(u = S/J\)-fixed

\[ E_0 = S + J + \frac{\bar{\lambda}}{2J} k^2 u (1 + u) - \frac{\bar{\lambda}^2}{8J^3} k^4 u (1 + u)(1 + 3u + u^2) + \frac{\bar{\lambda}^3}{16J^5} k^6 u (1 + u)(1 + 7u + 13u^2 + 7u^3 + u^4) + \mathcal{O}\left(\frac{1}{J^7}\right) \]

- two possible relations between \(\text{AdS}_5\) and \(\text{AdS}_4\) results

1) \(\bar{\lambda}_{\text{AdS}_5} \mapsto \bar{\lambda}_{\text{AdS}_4}\)

2) \(E_{\text{AdS}_5} \mapsto 2E_{\text{AdS}_4}, J_{\text{AdS}_5} \mapsto 2J_{\text{AdS}_4}, \bar{\lambda}_{\text{AdS}_5} \mapsto 4\bar{\lambda}_{\text{AdS}_4}\)
Quantum corrections:

**V1.** Hamiltonian formalism; works great with static gauge $t = \kappa \tau$

\[ E = \frac{1}{\kappa} \langle \psi | H | \psi \rangle \rightarrow E_1 = \frac{1}{\kappa} \langle \psi | H_2 | \psi \rangle \]

\[ E_1 = \frac{1}{2\kappa} \sum_{n=-\infty}^{\infty} (-)^{F_i \omega_{n,i}} \leftarrow \text{fluctuation frequencies} \]

**V2.** Lagrangian formalism in conformal gauge

Large charges $\rightarrow$ the partition function localizes around a single critical point of the action; correction to energy from free energy while accounting for renormalization of the other charges

\[ E_1 \propto \ln \text{sdet}K \rightarrow E_1 = \frac{1}{2\kappa} \sum_{n=-\infty}^{\infty} (-)^{F_i \omega_{n,i}} \]

◊ carries over to higher loops
Quantum corrections:

- detailed knowledge of quadratic part of the action
  - from all-order action based on $OSp(6|4)/SU(3) \times U(1) \times SO(3, 1)$
    
    Arutyunov, Frolov; Stefanski; Fre, Grassi

  - used by Alday, Arutyunov, Bykov; Krishnan for SFS

  - General $\kappa$-symmetric form implying linearized sugra constraints

\[
L_{2F} = i(\eta^{ab}\delta^{IJ} - \epsilon^{ab}s^{IJ})\bar{\theta}^I \gamma_a D_J^{JK} \theta^K
\]

\[
D_b = \partial_b + \frac{1}{4} \partial_b X^M \omega_M^{AB} \Gamma_{AB}
\]

\[
D_J^{JK} = D_b \delta^{JK} - \frac{1}{8} \partial_b X^M E_M^A H_{ABC} \Gamma^{BC}(\sigma_3)^{JK}
\]

\[
+ \frac{1}{8} e^\phi \left[ F(0)(\sigma_1)^{JK} + \Phi(2)(i\sigma_2)^{JK} + \Phi(4)(\sigma_1)^{JK} \right] \gamma_b
\]

- In special Lorentz frame $\mathbb{CP}^3$ spin connection is not important

- After appropriate rotations projector is exposed; fix $\kappa$-symmetry

  e.g. SFS: $L = i\bar{\Psi}(\eta^{ab} - \epsilon^{ab}\Gamma_{11})(\tau_a \gamma_b + \tau_a \gamma^b)\Psi$, $\Psi = S^{-1}\theta$, $S = \exp(\kappa/2\sigma^3 \Gamma_{a3})$
Spectrum of quadratic fluctuations; spinning folded string
($\Phi = \bar{\Phi} + \epsilon \tilde{\Phi}$ and rotation on $\tilde{\Phi}$)

**Bosons:**
- two massless modes (one in AdS$_4$; one in $\mathbb{CP}^3$); canceled by ghosts
- three modes from AdS$_4$

$$\omega_{\pm}(n) = \sqrt{n^2 + 2\kappa^2 \pm 2\sqrt{\kappa^4 + n^2\nu^2}}$$
$$\omega_T(n) = \sqrt{n^2 + 2\kappa^2 - \nu^2}$$

- one-four modes from $\mathbb{CP}^3$ (reflects breaking $SO(6) \rightarrow SO(4)$)

$$\omega_H(n) = \sqrt{n^2 + \nu^2}$$
$$4 \text{ of } \omega_L(n) = \sqrt{n^2 + \frac{1}{4}\nu^2}$$

**Fermions:** (reflects breaking $SO(6) \rightarrow SO(4)$)

$$\omega_{\pm12}(n) = \frac{\nu}{2} + \sqrt{n^2 + \kappa^2}$$
$$\omega_{\pm34}(n) = \frac{1}{\sqrt{2}} \sqrt{n^2 + 2\kappa^2 \pm \sqrt{\kappa^4 + 4n^2\nu^2}}$$

$$e(n) = \omega_+ + \omega_- + \omega_T + \omega_H + 4\omega_L - \sum_{i=1}^4 (\omega_{+i} + \omega_{-i})$$
$$E_1 = \sum_n e(n)$$
\[ e(n) = \omega_+ + \omega_- + \omega_T + \omega_H + 4\omega_L - \sum_{i=1}^{4} (\omega_{+i} + \omega_{-i}) \quad E_1 = \int_0^\infty dpe(\kappa p) \]

a) \((S, J = 0)\): 
\[
E_1 = -\frac{5\ln 2}{2\pi} \ln S + \mathcal{O} \left( \ln^0 S \right)
\]

b) \((S, J \neq 0)\) 
\[
\left( u = \frac{l}{\sqrt{1+l^2}} \quad l = \frac{J}{\sqrt{\lambda_{\text{AdS}_4} \ln S}} \right)
\]
\[
E_1 = \frac{\nu}{2u} \left[ -(1-u^2) + \sqrt{1-u^2} - 2u^2 \ln u \right.

\[
- (2-u^2) \ln \left( \sqrt{2-u^2(1+\sqrt{1-u^2})} \right) + 2(1-u^2) \ln 2 \] \]

- contrast with AdS$_5 \times$S$^5$ energy shift

a) \((S, J = 0)\): 
\[
E_1 = -\frac{3\ln 2}{2\pi} \ln S + \mathcal{O} \left( \ln^0 S \right)
\]

b) \((S, J \neq 0)\) 
\[
\left( u = \frac{l}{\sqrt{1+l^2}} \quad l = \frac{J}{\sqrt{\lambda_{\text{AdS}_5} \ln S}} \right)
\]
\[
E_1 = \frac{\nu}{2u} \left[ -(1-u^2) + \sqrt{1-u^2} - 2u^2 \ln u \right.

\[
- (2-u^2) \ln \left( \sqrt{2-u^2(1+\sqrt{1-u^2})} \right) \] \]
Spectrum of quadratic fluctuations; circular rotating string
(\(\Phi = \tilde{\Phi} + \epsilon \tilde{\Phi}\) and rotation on \(\tilde{\Phi}\))

McLoughlin, RR, Tseytlin

- **Bosons:**
  - two massless modes (one in AdS\(_4\); one in \(\mathbb{C}P^3\)); canceled by ghosts
  - three modes from AdS\(_4\):
    \[
    \omega_T(n) = \sqrt{p_1^2 + \kappa^2} \quad \& \quad \text{two solutions of}
    \frac{1}{4}(\omega(n)^2 - n^2)^2 + r_1^2 \kappa^2 \omega(n)^2 - \left(1 + r_1^2\right) \left(\sqrt{\kappa^2 + k^2 \omega(n)} - kn\right)^2 = 0
    \]
  - one + four modes from \(\mathbb{C}P^3\) (reflects breaking \(SO(6) \rightarrow SO(6)\))
    \[
    \omega_H(n) = \sqrt{n^2 + (\omega^2 - m^2)} \quad \text{4 of} \quad \omega_L(n) = \sqrt{n^2 + \frac{1}{4}(\omega^2 - m^2)}
    \]

- **Fermions:** (reflects breaking \(SO(6) \rightarrow SO(6)\))
  \[
  \omega_{\pm 12}(n) = \pm \frac{\sqrt{r_1^2 \kappa m}}{2(\omega^2 + r_1^2 \kappa^2)} + \sqrt{(p_1 \pm b)^2 + (\omega^2 + k^2 r_1^2)} \quad b = -\frac{\kappa m}{\omega} \frac{\omega^2 - \omega^2}{2(m^2 + r_1^2 \kappa^2)}
  \]
  \[
  (\omega(n)^2 - n^2)^2 + r_1^2 \kappa^2 \omega(n)^2 - \left(1 + r_1^2\right) \left(\sqrt{\kappa^2 + k^2 \omega(n)} - kn\right)^2 = 0
  \]
  \[
  e(n) = \omega_+ + \omega_- + \omega_T + \omega_H + 4 \omega_L - \sum_{i=1}^{4} (\omega_+ + \omega_-) \quad E_1 = \frac{1}{2\kappa} \sum_n e(n)
  \]
• Scaling limit: \( S, J \to \infty \) with fixed \( u = S/J \); expand in \( 1/J \)
  
  ▪ features similar to AdS\(_5 \times S^5\) calculation: sum at finite \( J \) and \( S \) is convergent but some terms in the expansion lead to divergent contributions

  ▪ e.g. leading term in the scaling limit (\( J = \sqrt{\lambda} \omega, \ n = \omega x \))

\[
e^{\text{sum}}(n) = \frac{1}{2\omega} \left[ n \left( 3n - 4\sqrt{n^2 + k^2u(1+u)} + \sqrt{n^2 + 4k^2u(1+u)} \right) \right. \\
\left. - k^2(1+u)(1+3u) \right] + O \left( \frac{1}{\omega^3} \right)
\]

\[
e^{\text{int}}(x) = \frac{k^2(1+u)}{2\omega} \left[ \frac{1+u(3+2x^2)}{(1+x^2)^{3/2}} - 2 \frac{1+u(3+8x^2)}{(1+4x^2)^{3/2}} \right] + O \left( \frac{1}{\omega^3} \right)
\]

▪ \( e^{\text{int}}(0) = e^{\text{sum}}(0) \to \) ignore last term in \( e^{\text{sum}}(n) \) and replace its contribution with the integral of \( e^{\text{int}}(x) \); resummation of divergences

▪ Direct numerical evaluation confirms this interpretation
• Scaling limit: \( S, J \to \infty \) with fixed \( u = S/J \); expand in \( 1/J \)

  - features similar to AdS\(_5\times S^5\) calculation: sum at finite \( J \) and \( S \) is convergent but some terms in the expansion lead to divergent contributions

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e_{\text{sum}}(n) = \frac{1}{2\omega} \left[ n \left( 3n - 4\sqrt{n^2 + k^2u(1+u)} + \sqrt{n^2 + 4k^2u(1+u)} \right) \right. \\
\left. - k^2(1+u)(1+3u) \right] + \mathcal{O} \left( \frac{1}{\omega^3} \right)
\]

\[
e_{\text{int}}(x) = \frac{k^2(1+u)}{2\omega} \left[ \frac{1+u(3+2x^2)}{(1+x^2)^{3/2}} - 2 \frac{1+u(3+8x^2)}{(1+4x^2)^{3/2}} \right] + \mathcal{O} \left( \frac{1}{\omega^3} \right)
\]

- \( \sum_{n} \quad \Rightarrow \quad \omega \int_{-\infty}^{+\infty} dx \Rightarrow \{ e_{\text{sum}} \ e_{\text{int}} \} \) are expansions in \{ \( 1/J \)\( ^{\text{even}} \) \( 1/J \)\( ^{\text{odd}} \) \)

→ analyze separately
\[ E_{1}^{\text{odd}} = \frac{\omega}{2\kappa} \int_{-\infty}^{\infty} dx \ e_{\text{reg}}^{\text{int}}(x) \]
\[ = - \frac{\bar{\lambda}^{1/2}k^2}{J} \ln 2 \ u(1 + u) + \frac{\bar{\lambda}^{3/2}k^4}{2J^3} \ln 2 \ u(1 + u)(1 + 3u + u^2) \]
\[ - \frac{\bar{\lambda}^{5/2}k^6}{8J^5} u(1 + u) \left[ 3(1 + 7u + 13u^2 + 7u^3 + u^4) \ln 2 \right] \]
\[ + \frac{\bar{\lambda}^{5/2}k^6}{6J^5} u^3(1 + u)^3 + \mathcal{O}\left(\frac{1}{J^7}\right) \]

- combine with leading order terms

\[ E_0 + E_{1}^{\text{odd}} = S + J + \frac{\bar{h}^2(\bar{\lambda})k^2}{2J} u(1 + u) - \frac{\bar{h}^4(\bar{\lambda})k^4}{8J^3} u(1 + u)(1 + 3u + u^2) \]
\[ + \frac{\bar{h}^6(\bar{\lambda})k^6}{16J^5} u(1 + u)(1 + 7u + 13u^2 + 7u^3 + u^4) \]
\[ + \frac{\bar{h}^5(\bar{\lambda})k^6}{6J^5} u^3(1 + u)^3 + \mathcal{O}\left(\frac{1}{J^7}\right) \]

- introduce \( \bar{h}(\bar{\lambda}) = \sqrt{\bar{\lambda}} - \ln 2 + \mathcal{O}\left(\frac{1}{\sqrt{\bar{\lambda}}}\right) \); to this order \( \bar{h}(\bar{\lambda})^n \) contributes the first two terms in its expansion
combine with leading order terms $\bar{h}(\bar{\lambda}) = \sqrt{\bar{\lambda}} - \ln 2 + \mathcal{O}\left(\frac{1}{\sqrt{\bar{\lambda}}}\right)$

\[
(E_0 + E_1^{\text{Odd}})_{\text{AdS}_4 \times \text{CP}^3} = S + J + \frac{\bar{h}^2(\bar{\lambda}) k^2}{2J} u(1 + u) - \frac{\bar{h}^4(\bar{\lambda}) k^4}{8J^3} u(1 + u)(1 + 3u + u^2) \\
+ \frac{\bar{h}^6(\bar{\lambda}) k^6}{16J^5} u(1 + u)(1 + 7u + 13u^2 + 7u^3 + u^4) \\
+ \frac{\bar{h}^8(\bar{\lambda}) k^8}{6J^7} u^3(1 + u)^3 + \mathcal{O}\left(\frac{1}{J^7}\right)
\]

\[
(E_0 + E_1^{\text{Odd}})_{\text{AdS}_5 \times S^5} = J + S + \frac{\lambda_{\text{AdS}_5} k^2}{2J} u(1 + u) - \frac{\lambda_{\text{AdS}_5}^2 k^4}{8J^3} u(1 + u)(1 + 3u + u^2) \\
+ \frac{\lambda_{\text{AdS}_5}^3 k^6}{16J^5} u(1 + u)(1 + 7u + 13u^2 + 7u^3 + u^4) \\
+ \frac{\lambda_{\text{AdS}_5}^5/2 k^6}{16J^5} u(1 + u)(1 + 7u + 13u^2 + 7u^3 + u^4) \\
+ \frac{\lambda_{\text{AdS}_5}^5/2 k^6}{3J^5} u^3(1 + u)^3 + \mathcal{O}\left(\frac{1}{J^7}\right)
\]

◊ The map: $E_{\text{AdS}_5} \mapsto 2E_{\text{AdS}_4}$, $J_{\text{AdS}_5} \mapsto 2J_{\text{AdS}_4}$, $\bar{\lambda}_{\text{AdS}_5} \mapsto 4\bar{h}^2(\bar{\lambda}_{\text{AdS}_4})$

after all parameters of the solution are expressed in terms of charges!
(\bar{E}_1^{\text{even}})_{\text{AdS}_4 \times \mathbb{CP}^3} = \frac{1}{\kappa} \sum_{n=1}^{\infty} \epsilon_{\text{reg}}(n)

= \frac{-\bar{\lambda}k^4(1+u)^2u^2}{2^3J^2} \left( 6\zeta(2) - 15k^2u(1+u)\zeta(4) + \frac{315}{8}k^4u^2(1+u)^2\zeta(6) + \ldots \right)
+ \frac{\bar{\lambda}^2k^6(1+u)^2u^2}{2^6J^4} \left( 24(1+2u-u^2)\zeta(2) + 15k^2u^2(1+u)(5+13u)\zeta(4) 
- \frac{63}{2}k^4u^2(1+u)^2(5+22u+27u^2)\zeta(6) + \ldots \right) + O\left(\frac{1}{J^6}\right)

(\bar{E}_1^{\text{even}})_{\text{AdS}_5 \times S^5} = \frac{1}{\kappa} \sum_{n=1}^{\infty} \epsilon_{\text{reg}, \text{AdS}_5 \times S^5}(n)

= \frac{-\lambda k^4(1+u)^2u^2}{2^2J^2} \left( 4\zeta(2) - 8k^2u(1+u)\zeta(4) + 20k^4u^2(1+u)^2\zeta(6) + \ldots \right)
+ \frac{\lambda^2k^4(1+u)^2u^2}{2^5J^4} \left( 16k^2(1+2u-u^2)\zeta(2) + 8k^4u^2(1+u)(5+13u)\zeta(4) 
- 16k^6u^2(1+u)^2(5+22u+27u^2)\zeta(6) + \ldots \right) + O\left(\frac{1}{J^6}\right)

- Besides \ E_{\text{AdS}_5} \mapsto 2E_{\text{AdS}_4}, \ J_{\text{AdS}_5} \mapsto 2J_{\text{AdS}_4}, \ \bar{\lambda}_{\text{AdS}_5} \mapsto 4\bar{h}^2(\bar{\lambda}_{\text{AdS}_4}), \ mapping \ the \ two \ expressions \ into \ each \ other \ requires \ a \ re-identification \ of \ zeta-constants; \ \textbf{physically unjustified} \ \mapsto \ \textbf{differs \ from \ proposed \ BA}
Conclusions

- The natural worldsheet and (built in) Bethe Ansatz regularization schemes are not necessarily the same
- Magnon dispersion relation receives (in conformal gauge) scheme-dependent corrections
- Remains an open question whether all quantities depend only on $\bar{h}(\lambda)$; if so, choose some anomalous dimension as physical coupling
- conjectured all-loop Bethe Ansatz reproduces (the continuous) part of the worldsheet results; finite size effects need more analysis
  - giant magnon finite size effects seem to work out fine, however
  - Grignani, Harmark, Orselli, Semenoff; Bombardelli, Fioravanti; Lukowski, Sax; Ahn, Bozhilov;...
- Possible origin of differences
  - misidentification of sectors/excitations
  - misidentification of S-matrix, especially $S_{AB}$
  - breakdown of integrability at the quantum level