Consistency of $\mathbb{Z}_n$ orbifold conformal field theories on K3

Katrin Wendland
UNC-CH

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$M$: moduli space of $\mathcal{N}=(4,4)$ SCFTs with $c=6$

Local structure
[Narain '86; Seiberg '88; Cecotti '91]

Global structure
[Narain '86; Aspinwall/Morrison '94; Nalum/W. '99]

Special points and subvarieties
- Gepner models
- Nonlinear $\sigma$-models
  - Toroidal theories
  - Orbifold CFTs

-1-
Local description of the moduli space

\[ \mathfrak{F}^{4,4+5} = O^+(4,4+5;\mathbb{R})/SO(4) \times O(4+5) \]
Grassmannian of oriented positive definite fourplanes in \( \mathbb{R}^{4,4+5} \)

\[ O^+(a,b;\mathbb{R}) \subset O(a,b;\mathbb{R}) \] components containing \( SO(a) \times O(b) \)

modular properties of conformal field theoretic elliptic genus \( E^5 \)

\[ [\text{Nahm 14.39}] \Rightarrow \downarrow \Rightarrow [\text{Schellekens/Warner 86}] \]

\[ \begin{cases} \text{d=0} \quad \text{or} \quad \{ \text{d=16} \\ \text{torus } X \} \\ \text{K3 surface } X \end{cases} \]

\[ M = M^{\text{torus}} \cup M^{K3} \]

\[ E^5 = E_x \text{ elliptic genus of } X \]

\[ [\text{Eguchi/Ooguri/ Taormina/ Yang 89}] \]

-2-
Local structure

\[ O^+ (H^{even} (X, \mathbb{R})) / SO(4) \times O(4+\delta) = J^{4+2+\delta} \cong J^{\delta} \times \mathbb{R}^+ \times H^2 (X, \mathbb{R}) \]

\[ x \mapsto (\xi, \nu, B) \]

\[ x = \text{span}_\mathbb{R} \left( \xi (\nu), \nu + B + (\nu - \frac{2i}{\xi}) \nu \right) \]

\[ \xi (\nu) = \nu - \langle \xi, B \rangle \nu \quad \text{for} \quad \perp \nu, \nu^0 \]

\( \nu, \nu^0 \): primitive null vectors, \( \langle \nu, \nu^0 \rangle = 1 \)

In the even self-dual lattice \( H^{even} (X, \mathbb{Z}) \)

\( \nu, \nu^0 \) generate \( H^2 (X, \mathbb{R}) \), \( H^0 (X, \mathbb{Z}) \)

Global structure

\[ M = O^+ (H^{even} (X, \mathbb{Z})) \backslash O^+ (H^{even} (X, \mathbb{R})) / SO(4) \times O(4+\delta) \]

- "classical symmetries" which fix \( \nu, \nu^0 : O^+ (H^2 (X, \mathbb{Z})) \)
- shifts of \( B \) by \( 2 \xi H^2 (X, \mathbb{Z}) \): \( (\nu, \nu^0) \rightarrow (\nu, \nu^0 + 2 - \frac{2i}{\xi} \nu) \)
- Fourier-Mukai transform \( \nu \mapsto \nu^0 \)

[Aspinwall/Morrison '94]

[Narain '96; Aspinwall/Morrison '94]

[Nahm/Mü '99]

-3-
# $\mathbb{Z}_m$ orbifold constructions of K3

<table>
<thead>
<tr>
<th>$\mathbb{Z}_m$ type</th>
<th>Fixed pts.</th>
<th>$\mathrm{exc. div.}$</th>
<th>$h^0(T^4/\mathbb{Z}_m)$</th>
<th>Kummer type lattice</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}_4$</td>
<td>$16 \mathbb{Z}_2$ type</td>
<td>${E_i/\iota_0\mathbb{Z}_4^i}$</td>
<td>$4 + 16 \cdot 1 = 20$</td>
<td>$\Pi = \text{span}<em>\mathbb{Z} { E</em>{2i}, \mathbb{Z}/2 \mathbb{Z}_2 }$</td>
</tr>
<tr>
<td>$\mathbb{Z}_3$</td>
<td>$9 \mathbb{Z}_3$ type</td>
<td>${E_i^{(1)}/\iota_1\mathbb{Z}_3^i}$</td>
<td>$2 + 9 \cdot 2 = 20$</td>
<td>$\Pi = \text{span}<em>\mathbb{Z} { E</em>{i}, \frac{1}{2} (E_{i}^- - E_{i}^+), \text{I} }$</td>
</tr>
<tr>
<td>$\mathbb{Z}_2$</td>
<td>$1 \mathbb{Z}_2$ type</td>
<td>${E_i/\iota_1\mathbb{Z}_2^i}$</td>
<td>$2 + 4 \cdot 3 + 6 \cdot 1 = 20$</td>
<td>$\Pi = \text{span}<em>\mathbb{Z} { E</em>{1}, 3E_{1}^{(1)} + 2E_{1}^{(2)} + E_{1}^{(3)}, \text{I} }$</td>
</tr>
</tbody>
</table>

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**Notation:**
- $E_i$ denotes a generator of $E_i$.
- $E_{i}^- - E_{i}^+$ represents a hyperplane.
- $\Pi$ is the Kummer type lattice.
- $\mathbb{Z}/2 \mathbb{Z}_2$ indicates the action of $\mathbb{Z}_2$.
- $\frac{1}{2} (E_{i}^- - E_{i}^+)$ corresponds to parallel planes.
- $3E_{1}^{(1)} + 2E_{1}^{(2)} + E_{1}^{(3)}$ is a combination of generators.
- $\text{I}$ represents an invariant.

**Details:**
- The $\mathbb{Z}_m$ orbifold constructions are studied for different $\mathbb{Z}_m$ types, with fixed points and exceptional divisors given.
- The $h^0(T^4/\mathbb{Z}_m)$ values indicate the number of holomorphic sections of the line bundle over $T^4/\mathbb{Z}_m$.
- The Kummer type lattice $\Pi$ is constructed using generators and relations involving $E_i$ and $\mathbb{Z}/2 \mathbb{Z}_2$ actions.

**Example:**
- For $\mathbb{Z}_4$, $16 \math{Z}_2$ type, the fixed points are $E_i/\iota_0\mathbb{Z}_4^i$, and the $h^0(T^4/\mathbb{Z}_4)$ is $4 + 16 \cdot 1 = 20$.
- The Kummer type lattice $\Pi$ involves $E_{2i}$ and includes $\mathbb{Z}/2 \mathbb{Z}_2$ hyperplane.

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**Mathematical Expressions:**
- $E_i^{(1)}$ denotes a specific configuration of $E_i$.
- $\frac{1}{2} (E_{i}^- - E_{i}^+)$ signifies a combination that forms parallel planes.
- $3E_{1}^{(1)} + 2E_{1}^{(2)} + E_{1}^{(3)}$ is a linear combination of $E_i$.
- $\text{I}$ is an invariant that remains unchanged under the orbifold action.

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**Additional Notes:**
- The constructions involve detailed algebraic and geometric considerations.
- The orbifold $T^4/\mathbb{Z}_m$ is studied for different values of $m$.
- The lattice $\Pi$ is constructed to reflect the orbifold structure.

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**Conclusion:**
- The study of $\mathbb{Z}_m$ orbifold constructions of K3 provides insights into the geometric and algebraic properties of these spaces.
- The lattice $\Pi$ is crucial in understanding the structure and symmetry of the orbifold.

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**Further Reading:**
- Study of orbifolds in algebraic geometry.
- Detailed analysis of Kummer surfaces.
- Applications in string theory and mirror symmetry.
**Result**

Consistency of $\mathbb{Z}_m$ orbifold conformal field theories enforces

$$x_T = \text{span}_R \left( \frac{1}{2} \langle \Sigma \rangle, v^0 + B_T + (v_T - \frac{27}{4}) v \right)$$

$$\pi_\Sigma x_T = \text{span}_R \left( \frac{1}{2} \langle \Sigma \rangle, \Sigma_0 + B + (v - \frac{3}{4}) \Sigma \right)$$

where

$$\Sigma = \pi_\Sigma \Sigma_T$$

$$B = \frac{1}{m} B_T + \frac{1}{m} B_2^{(m)}$$

$$\langle B_2^{(m)}, E \rangle = \frac{m}{3}$$ for $E$ corresponding to a $\mathbb{Z}_m$-type fixed point

$$V = \frac{v_T}{m}$$

see also: $\mathbb{Z}_3$: [Aspinwall '95]

[Douglas '86; Blum/Intriligator '97]